

MU120 Unit 14



The Open
University

Mathematics
and Computing
A first level
multidisciplinary
course

Open Mathematics

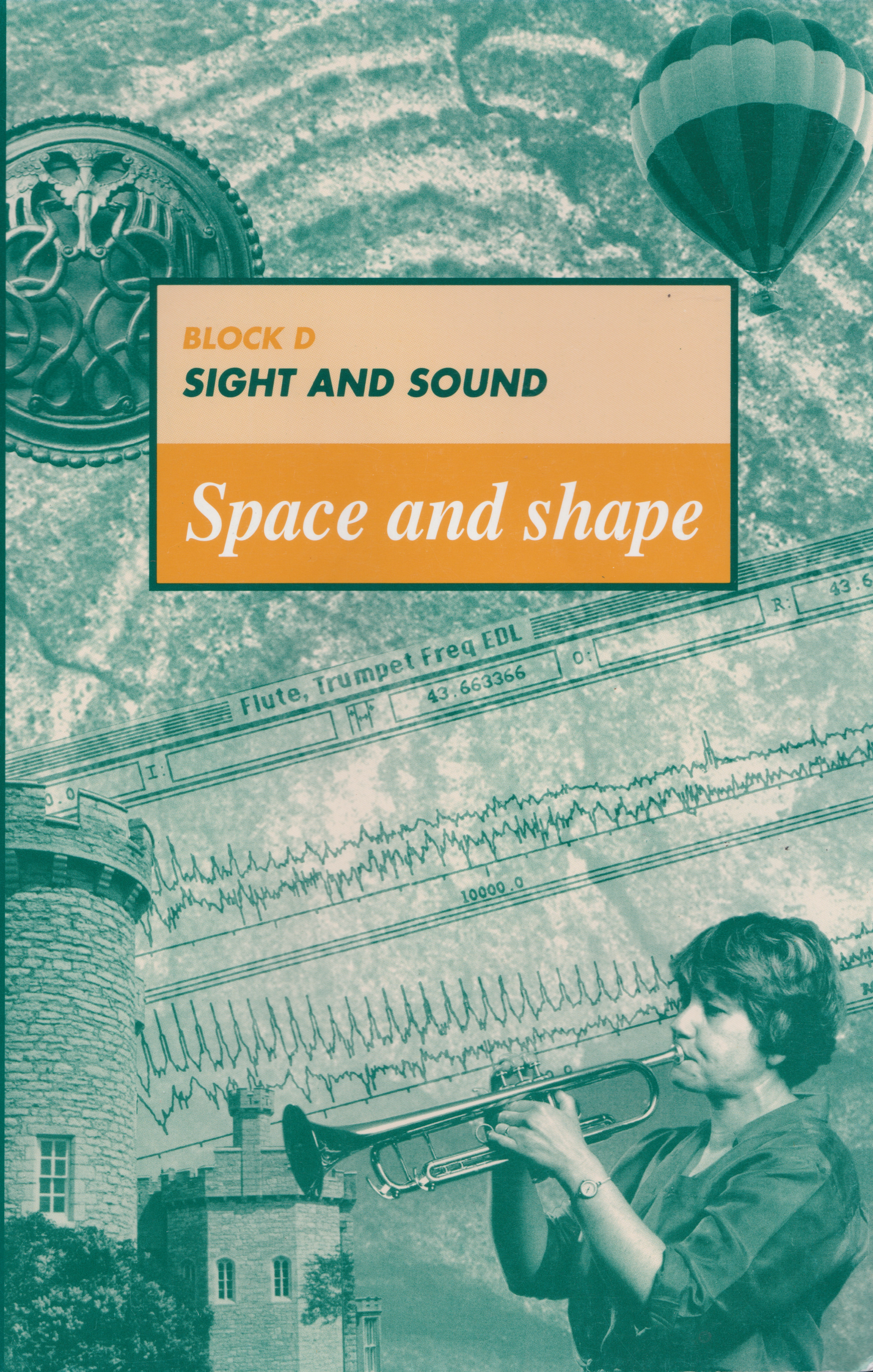
UNIT

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BLOCK D

SIGHT AND SOUND

Space and shape





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Study guide

Unit 14 marks the beginning of the final block. It revisits some ideas about maps that you first met in *Unit 6*, and uses them to motivate an introduction to some geometric and trigonometric ideas, both in the plane and on the sphere.

Section 1 asks you to view a videotape band and work through an example and some activities based on it. The videotape band is entitled ‘Showing the way they went’, and looks back at the making of the videotape band you watched in *Unit 6*, about planning a walk. You will need the OS map from *Unit 6* and the Learning File notes you made while studying that unit. In order to study Section 1, you will *need* to view the videotape band; however, if it is inconvenient to do so as you start your study of *Unit 14*, you could go straight on to Section 2, coming back to Section 1 later.

Section 2 looks again at notions of similarity of geometric figures (first discussed in *Unit 2*), in terms of relationships among lengths, angles and areas. It also revisits Pythagoras’ theorem. No course components other than the main text are involved.

Section 3 explores some aspects of the trigonometry of triangles in the plane, as well as a discussion of π . This section also draws on a brief *Calculator Book* section, which provides a program to implement the cosine formula, as well as an audiotape band and a short extract from a Sherlock Holmes story in the reader, both partly about the uses of trigonometry. If it is inconvenient for you to listen to the audiotape band during your study of Section 2, you could postpone it until later.

Section 4 is mathematically more challenging in places, as it develops formulas and ideas for calculating distances, not in the plane, but on the surface of a sphere. Again, there is a short *Calculator Book* section, implementing the distance formulas that the section develops.

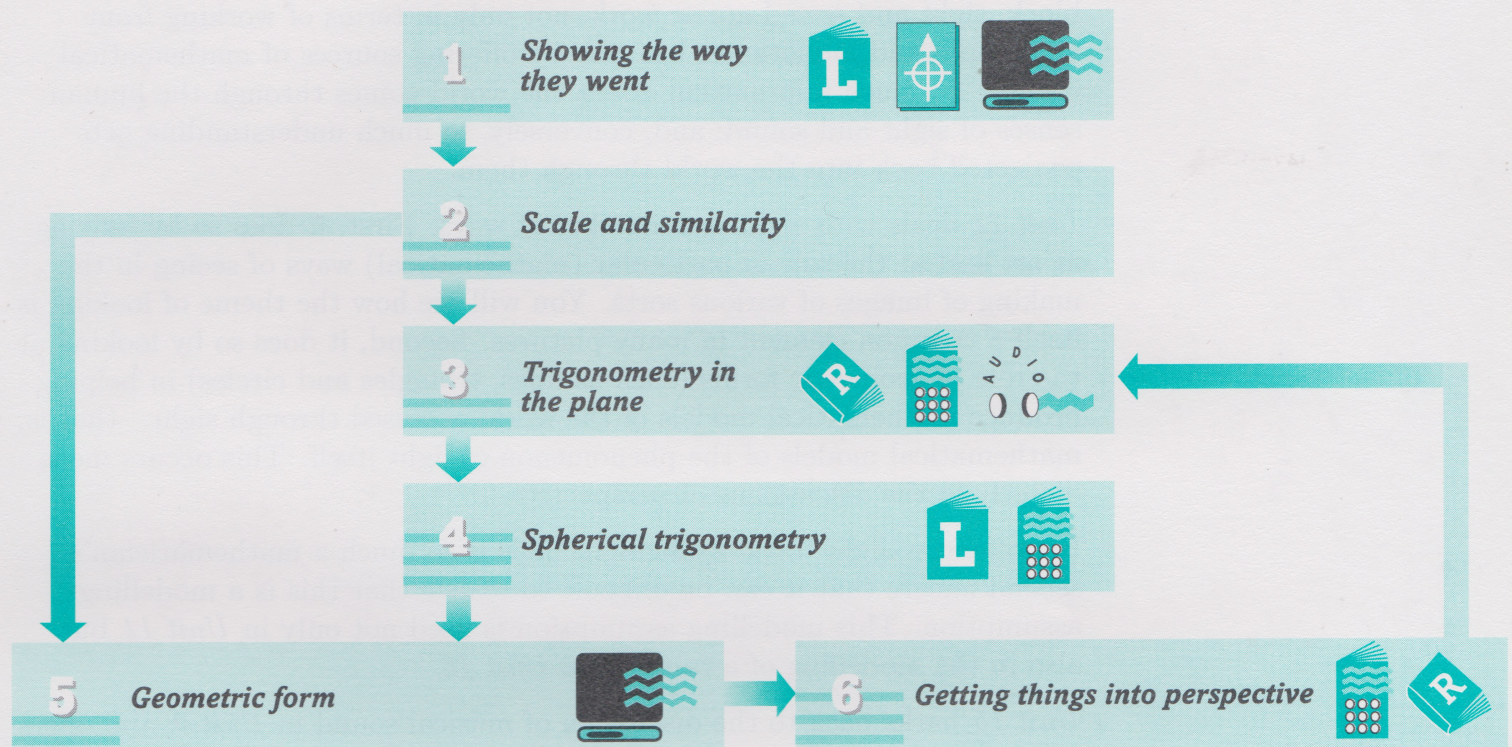
Section 5 is more unusual in form. It broadens the discussion to look at the use of geometric ideas in art, but also starts to explore the nature of geometric thinking and imaging. In short, the theme is one of how to see the world geometrically. The central focus is a videotape band, entitled ‘Geometric form: in the eye of the beholder’, which you may well want to watch more than once.

Section 6 takes some of the ideas on perspective from Section 5, and offers a mathematical discussion of them. This section draws on another short *Calculator Book* section and two reader articles.

There is an *optional* Appendix that derives some of the formulas discussed in Sections 3 and 4.

The television programme *Building by Numbers* is very relevant to this unit, especially to Sections 5 and 6.

There are Handbook sheets associated with this unit, for you to record any new terms or techniques. There are also Learning File sheets for certain activities.



Summary of sections and other course components needed for *Unit 14*

Introduction to Block D

This final block of the course is entitled *Sight and Sound*. Throughout the block, sight and sound are at work, not only in terms of working from videotapes and audiotapes, but also by offering sources of mathematical insight. So much information about the world comes through the human senses of sight and sound; and, conversely, so much understanding gets projected back into the world through them.

Unit 14 deals with sight in a number of ways. First, it does so by looking at art and at the role of particular (mathematical) ways of seeing in the making of images of various sorts. You will see how the theme of looking is itself a common element in many pictures. Second, it does so by looking at the role of geometric forms (such as lines, triangles and circles) in helping produce mathematical models of the world accessed through sight—that is, mathematical models of the phenomenon of light itself. This occurs most directly in the discussions of perspective.

Representing light ‘rays’ by straight lines is so much a mathematician’s second nature that it can be difficult to realize that this is a modelling assumption. This modelling assumption is used not only in *Unit 14* but also in the modelling of a rainbow in *Unit 16*.

Unit 15 harks back to the discussion of musical sound in *Unit 9*, and offers a more mathematical and technological look at characteristics of sound of various sorts. Ideas of trigonometric relationships, alluded to in *Unit 9*, come to the fore, and links are made to the trigonometric computations and ideas from *Unit 14*.

All five senses allow humans to gain information about the material world that surrounds them, and also to project understanding onto that world. Mathematical ideas frequently structure such interactions. As you study this block, you may care to keep an eye out for the place and role of mathematics in human interactions with the material world.

While there are some new mathematical themes in this block, the overall theme is one of consolidation of your learning—this is particularly true of *Unit 16*. With most of the course behind you, looking back and making links with previous course ideas and topics should be uppermost in your mind. This block also gives you the opportunity to ‘stand back’ and consider the ways you have been working on and learning mathematics. Identifying skills and strategies—ways of working—can be useful, especially when you find yourself faced with problems to solve. A main aim of this course is that you should be able to use what you have learned in other situations, and identifying what you have done and how you have done it is one way of helping you to achieve this aim.

Introduction

The problems of representing a three-dimensional world on a two-dimensional map were discussed in *Unit 6*. The properties of a good map are that it shows true *shape*, accurate *distances*, equal *areas* and consistent *orientation*. So good maps allow the computation of aspects of the represented situation by means of actions on the representation itself.

Paintings, like maps, provide a two-dimensional representation of a three-dimensional world. However, shapes are not generally preserved in them. A shape drawn ‘in perspective’ to ‘look like’ a square is not itself square, but instead a parallelogram. So making calculations and deductions from painted scenes that ‘look right’ presents problems.

Maps and paintings can be used to try to reconstruct the three-dimensional world they represent. But, however good the representation, there is always ambiguity in trying to reconstruct a three-dimensional object from a two-dimensional representation. Maps and paintings encode particular points of view, particular ways of seeing. The notion of viewpoint, and how it affects what is seen and shown, is an issue to bear in mind as you work through this unit.

Maps are examples of *visual* models of the real world. Mathematics is very much about using idealized models, and then inferring from these something about the real-world situation that is of interest. Among other things, this unit looks at the use of geometric ideas in making visual models of real-world situations, and then doing computations on these models as if they were the situations themselves.

One such visual model is the mathematical plane: a seamless expanse unlimited in all directions, the embodiment of uninterrupted flatness. It might have a distinguished point (labelled *O* for origin), and a pair of axes drawn at right angles, which allow arithmetic and algebra to be used, as well as giving a numerical pair of coordinates that uniquely label each point. The plane is the two-dimensional model of the real world that provides the basis for the discussions in Sections 2 and 3.

The ideas of Section 3 are extended in Section 4 to the sphere—a three-dimensional model of the world. Sections 5 and 6 discuss how the geometric ideas in Sections 2–4 are used in painting and architecture.

1 Showing the way they went



Aims This section aims to remind you of some of the ideas from earlier in the course, particularly ideas on maps from *Unit 6*. ◇

The ideas in this section are mostly revision, and will be consolidated and built upon as you progress through the unit.



Activity 1 Using your learning

- Look back at your Learning File notes for *Unit 6*. Should you need to, remind yourself about grid references, as well as how to convert map distances into ground distances (and vice versa) using the map scale.
- The video band you are about to watch begins by revisiting the walk from the *Unit 6* videotape band. Look now at the OS map and find Mam Tor (grid reference 127836), the beginning of the ridge along which the walkers went. From there, they walked to Hollins Cross, Back Tor and Lose Hill. Write down the grid references of these points and measure the distance on the map between each pair of adjacent points on the walk, and then convert these map distances to distances on the ground using the map scale.
- Now think about what is stressed and what is ignored by the OS map. How are the three-dimensional shape of the ridge and the features of the four points mentioned above conveyed on the map?



Now watch band 4 on Videotape 2. As you watch, think about how this journey would be represented on a map and in particular about which aspects this would stress and which aspects it could ignore.

Activity 2 Stressing and ignoring

- One of the pilots, Chris, noted his map position at different times along his journey, as shown in Table 1, in order to calculate his fuel consumption and hence estimate how much further he could go. Thus, his journey can be represented by a series of grid references. What does this representation stress and what does it ignore?

Table 1

Time	Grid reference	Notes
9.40	123825	Take off
10.26	157885	See other balloon land
10.53	184927	Point of no return
11.02	193940	Moors
11.14	209957	Moors
11.40	232998	Landing

- (b) Take a minute to think about the idea of stressing and ignoring. Have you found this a useful idea in helping you with your mathematical studies in this course?

Example 1 Reconstructing a balloon journey

The first balloon took off from grid reference 122825 and landed at 150888, taking about 45 minutes.

- (a) Find these two points on the map, and use direct measurement to find the total distance and bearing of the landing site from the take-off site.
- (b) Now use Pythagoras' theorem to find the total distance.
- (c) Hence find the average velocity (speed and direction) of the balloon, and compare it with the forecast wind velocities (mentioned at the planning meeting) of 8 km per hour at 25° at 2000 feet and 14 km per hour at 90° at 4000 feet.

Solution

- (a) Direct measurement on the map gives a distance of about 6.9 km. The grid direction is about 28° which, taking into account magnetic north, represents a bearing of $28^\circ + 5^\circ = 33^\circ$ on the ground.
- (b) Pythagoras' theorem can be used by separating the grid references into eastings and northings, as these represent the x - and y -coordinates in the plane. The squares of differences between the eastings and northings of the landing and take-off can be found, as in Table 2.

Table 2

	Grid reference	Easting	Northing
Take-off	122825	122	825
Landing	150888	150	888
Difference		$150 - 122 = 028$	$888 - 825 = 063$
Square of difference		784	3969

Then Pythagoras' theorem gives the map distance as $\sqrt{784 + 3969}$.

Each single grid reference unit represents 0.1 km, and so the distance in kilometres is:

$$0.1\sqrt{784 + 3969} \simeq 0.1 \times 68.9 = 6.9 \quad (\text{to two significant figures})$$

- (c) The journey took about 45 minutes, which gives an average speed of:

$$6.9 \text{ km} / 0.75 \text{ hours} = 9.2 \text{ km per hour}$$

So the average velocity of the balloon during flight was 9.2 km per hour at a bearing of 33° .

The weather forecast was for a wind velocity of 8 km per hour (at 25°) at 2000 feet and of 14 km per hour (at 90°) at 4000 feet. So you would expect the average velocity to be somewhere between 8 and 14 km per hour and at a bearing between 25° and 90° . The fact that the average velocity is much closer to the forecast for 2000 feet than to that for 4000 feet supports the video commentary that, although the wind velocity forecast at the lower altitude was good, the westerly wind forecast at 4000 feet did not materialize, causing the balloon to land further north than was originally intended.

Note that the data from the video indicate that the whole flight was not in the same direction, as the balloon flew at a number of different altitudes picking up different velocity winds. So comparing the average velocity with the forecast wind velocities at different heights is difficult.

Remember that, for velocity (discussed in *Unit 11*), both the magnitude (speed) and direction (bearing) are important. The direction as well as the speed of the wind and of the balloon are crucial in planning and navigating balloon flight.

Activity 3 Reconstructing Chris' balloon trip

Remember from the video that Chris recorded his map positions from his GPS (global positioning satellite) device and used this information to assess his fuel situation before passing the point of no return over the moors. Chris' grid references and times in Table 1 can be used to calculate the distances and speeds for the various legs of his trip, and so reconstruct one aspect of his journey. This is done for the first leg of his trip in Table 3. Complete Table 3 by calculating the distances and speeds for the second and third legs.

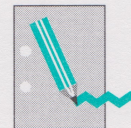
Table 3

Grid ref. point 1	Grid ref. point 2	Distance east	Distance north	Distance as crow flies (km)	Time taken (h)	Speed (km/h) (2 s.f.)
123825	157885	$157 - 123 = 034$	$885 - 825 = 060$	$0.1\sqrt{1156 + 3600} = 6.896$	$46/60 = 0.767$	9.0
157885	184927					
184927	193940					

Chris acquired the data giving his position directly from his GPS device. Although he did plot his position on the map, he did not need to do so in order to calculate the distances and speeds. You were able to estimate the distances of each leg of Chris' balloon trip without measuring anything on the map, but by using Pythagoras' theorem instead.

Activity 4 Auditing skills and strategies

As well as consolidating the mathematical ideas and techniques you have learned, this block gives you the opportunity to think about and use the different skills and strategies you have met. Indeed, you have just been involved in using a strategy—stressing and ignoring—to think about an OS map. As you work through this unit, and the remainder of the block, collect together the skills and strategies you have been using and identify an example where each has been useful. This will help you to pull together some of the ways you have been doing and learning mathematics.



Outcomes

After studying this section, you should be able to:

- ◇ use, with confidence, grid references, map bearings, map scales and Pythagoras' theorem (Activities 1, 2, 3).

2 Scale and similarity

Aims This section aims to explore the idea of similarity of geometric figures, and to look at various consequences for lengths and areas. ◇

2.1 Similar figures

Figure 1 is part of the OS map from *Unit 6*, which you used again in the previous section. It shows a number of fields whose boundaries, indicated by solid lines, are irregular in shape.

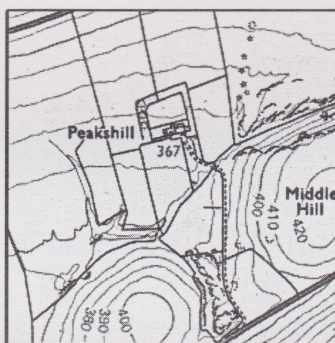


Figure 1

In each case, the map gives a true representation of the shape of the actual field on the ground (to within the limits imposed by printing, the difficulty of determining exactly where ‘the edge’ of the real field is, and so on—remember that a map is a mathematically structured image). It does so because, like the map as a whole, each field is drawn to scale.

The scale of the map is 1 : 25 000. Recall from *Unit 6* that, among other things, this means:

$$\text{distance on the ground} = 25\,000 \times \text{corresponding distance on the map}$$

In general, the relationship between distances on a map and corresponding distances on the ground is:

$$\text{distance on the ground} = \text{scale factor} \times \text{distance on the map}$$

Thus, the map represents distances accurately—and consistently. When you measure a distance on the map, your answer does not give the distance on the ground immediately, but only after you have carried out some mathematical processing (here, by multiplying it by 25 000).

From *Unit 13*, you will recognize this as a directly proportional relationship.

- There is one kind of measurement you can make on a map that does give the corresponding measurement on the ground directly—without any processing. What is it?

The map represents angles identically—when you measure (with a protractor) the angle between two lines on the map, your answer gives you immediately the corresponding angle that you would find on the ground. For instance, the angle on the map between the lines marking the boundary of a field at a corner is exactly the same as the angle between the corresponding stretches of the boundary of the field—marked, for example, by dry-stone walls—on the ground.

Thus, the basic assumptions underlying the use of maps are that you can work out lengths on the ground from measurements of lengths on the map; and that you can find angles directly by measuring angles on the map.

These properties are also true of geometric figures in the plane. If one figure is a scaled version of another, then the length of any side will be the same constant scale factor times the corresponding length in the other figure, and the corresponding angles of the two figures will be the same. Two such figures are called *similar*.

Similar figures have the same shape as each other, though they may well be of different sizes. Similarity is the mathematical formulation of the everyday notion of ‘same shape’. For instance, all circles are similar to one another; so are all squares. All equilateral triangles are similar to one another. However, for other figures, you have to think more carefully. Look at Figure 2.

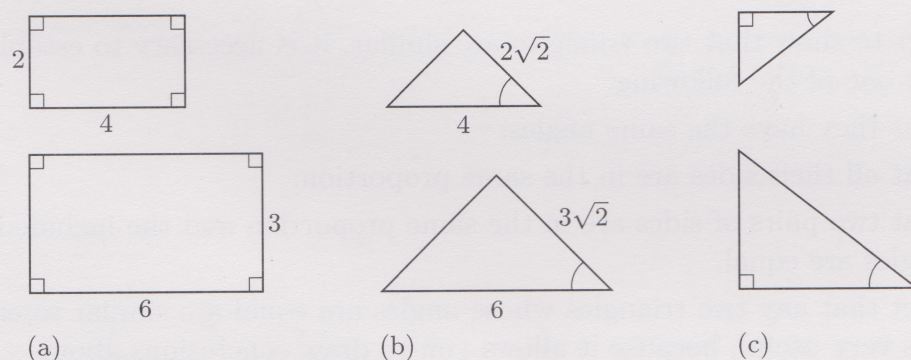


Figure 2 Similar figures

The two rectangles in Figure 2(a) are similar to one another, because both the length and the width of the larger one are the same scale factor (1.5) multiplied by the corresponding dimensions of the smaller and because all the angles are right angles. However, as you will shortly see, not all rectangles are similar to one other.

Note this is a different point from the issue of measuring *bearings* obtained by using a protractor on a map and then comparing them with the results of using a magnetic compass in the real world. See Unit 6, Section 3.

Geometric figures in the plane are frequently referred to as *plane figures*.

You met the idea of similar figures in the final section of Unit 2. You might care to look back at it now.

The two triangles in Figure 2(b) are also similar to one another. The lengths of the corresponding sides in both triangles are in the ratio 3 : 2, just like those of the rectangles. Since the lengths of only two pairs of sides are given, however, this is not by itself enough to show that the triangles are similar. In this case, though, the angle between those sides is also the same in each triangle (the way they are marked indicates that this is the case); and this additional piece of information together with the given length relationships shows that the two triangles must be similar. In general, two triangles are similar if two pairs of sides are in the same proportion and the *included angles*—the angles between the mentioned pairs of sides—are equal.

The point of this last example is that comparing lengths is not the only way of telling whether or not two figures are similar. Information about the angles can help as well. If two triangles are similar, then their corresponding angles are equal. It is an important fact about triangles that the converse is also true: if two triangles in the plane have equal angles, then they are similar; that is, they *must* have the same shape. Since the angles of any triangle always add up to 180° , or two right angles, it is actually enough to know that two pairs of angles in the triangles are equal: if this is so, the third pair of angles must necessarily be equal also.

The two triangles in Figure 2(c) have the same angles. Here, though, there is an additional complication: one is turned upside down relative to the other. Even so, they are still judged to be similar. Imagine cutting either of the triangles out of the page and turning it over: you would obtain two triangles with the same shape. The two triangles in the figure are therefore similar; the difference between them is expressed by saying that they have different *orientations*.

In order to show that two triangles are similar, it is necessary to establish at least one of the following:

- ◇ that they have the same angles;
- ◇ that all their sides are in the same proportion;
- ◇ that two pairs of sides are in the same proportion and the included angles are equal.

The fact that any two triangles whose angles are equal are similar to each other is very useful, because it allows you to draw conclusions about lengths from a knowledge of angles: that is to say, it relates lengths and angles. But a word of warning: do remember that the fact that equality of angles implies similarity is *not* true of figures other than triangles. For example, all rectangles have the same angles (all the angles of any rectangle are right angles, after all); but not all rectangles are similar to each other. The fact that you can tell whether or not two triangles are similar by comparing their angles, and that other classes of figures do not have this property, is one reason why triangles play such an important role in geometry.

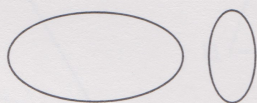
Activity 5 Similarity

Consider the pairs of figures in Figure 3. Identify those pairs in which the two figures are necessarily similar. (They have not been drawn accurately to scale, so this is not a test of your powers of pattern recognition, but of your ability to use the information given in the diagrams.)

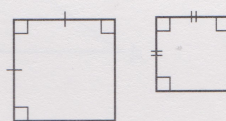
Some of the sides of some of the figures are marked with single or double tick marks, and some of the angles are marked with single or double arcs. In each part of Figure 3, features of figures which are marked in the same way are of the same size.



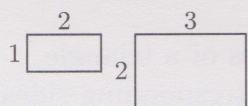
(a) Circles



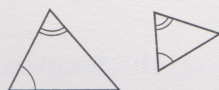
(b) Ellipses, with major axis twice the length of the minor axis



(c)



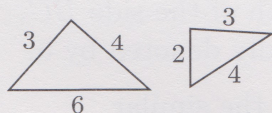
(d)



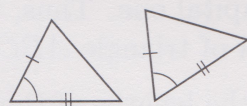
(e)



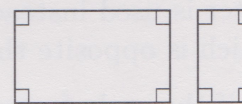
(f)



(g)



(h)



(i)

An *ellipse* is an elongated circle (or a squashed one, depending on your point of view). The direction of its greatest dimension is called its *major axis*, and the direction of its smallest dimension (which is at right angles to that of its greatest dimension) is called its *minor axis*.

Figure 3

Figure 4 (overleaf) shows two triangles that are similar to each other. Each triangle is therefore a scaled version of the other, and the two triangles have the same angles. The *vertices* (corners) of the two triangles have been labelled in such a way that the angles at A and A' are the same, the angles at B and B' are the same, and the angles at C and C' are the same. The vertices A and A' are said to be *corresponding vertices*, because their angles are the same; likewise B and B' are corresponding vertices, and so are C and C' . The side BC of triangle ABC and the side $B'C'$ of triangle $A'B'C'$ correspond, as do CA and $C'A'$, and AB and $A'B'$.

The fact that the triangles are similar implies the following relationship between their corresponding sides:

$$\begin{aligned} &\text{length of any side of one triangle} \\ &= \text{same scale factor} \times \text{length of corresponding side of the other} \end{aligned}$$

In other words, the lengths of corresponding sides of two similar triangles are directly proportional to each other.

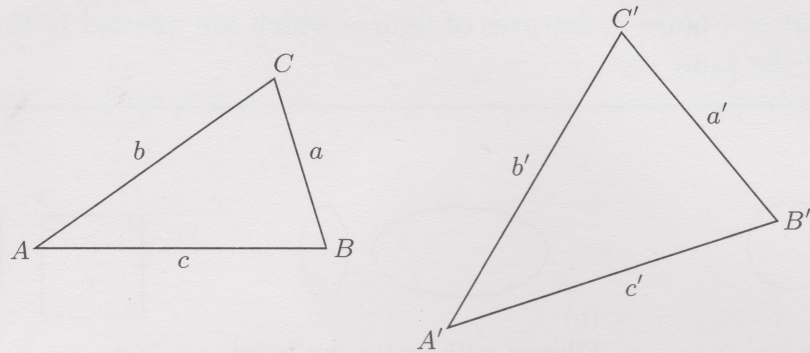


Figure 4

There is a standard way of writing the lengths of the sides of a triangle, which can be used to express the relationship between corresponding sides of similar triangles symbolically. The length of a side of a triangle is often denoted by the same letter as the label of the opposite vertex, but a small letter is used instead of the capital one. Thus, the length of the side BC , which is opposite the vertex A of triangle ABC , is usually denoted by a .

So if the scale factor relating the lengths of the sides of the similar triangles $A'B'C'$ and ABC is k , then:

$$a' = ka, \quad b' = kb, \quad c' = kc$$

To put the same thing slightly differently:

$$\frac{a'}{a} = \frac{b'}{b} = \frac{c'}{c} (= k, \text{ the scale factor})$$

That is to say, the ratios of the lengths of corresponding sides of two similar triangles are always the same. This is usually a mathematically more useful formulation than the one involving a scale factor, since it involves only the lengths of the sides of the triangles; it does not depend on knowing the scale factor.

The fact that triangles with the same angles are similar can sometimes be used to find the sizes of things that cannot be measured directly. Figure 5 shows a tree whose height is a matter of dispute.

► How can its height be measured?

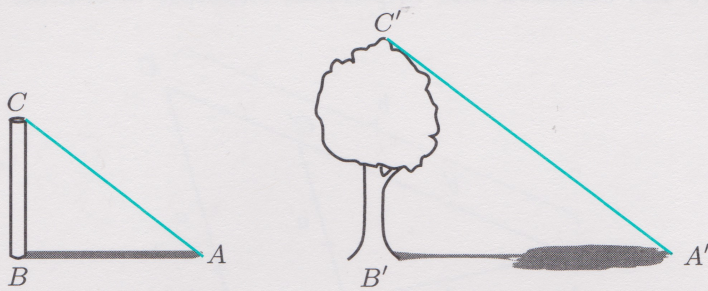


Figure 5 Finding the height of a tree

Provided that the day is sunny, it can be done by measuring the length of the tree's shadow, and then comparing it with the length of the shadow of a stick stuck upright in the ground, whose height is already known or can be measured. The sun's rays make the same angle with the ground whether they form the shadow of the tree or of the stick; both tree and stick are assumed to be vertical and the ground is assumed to be horizontal. It follows that the corresponding angles in the two triangles ABC and $A'B'C'$ are equal: the angles at A and A' are the same because they are both formed by the sun's rays; the angles at B and B' are both right angles because they are both the angle of a vertical to the horizontal ground; the angles at C and C' must then be the same because the angles of any triangle always add up to the same total of two right angles.

Triangles ABC and $A'B'C'$ are therefore similar to each other, and so:

$$\frac{a'}{a} = \frac{c'}{c}$$

Here a' is the height of the tree, which is unknown; a is the height of the stick, c' is the length of the tree's shadow, and c is the length of the shadow of the stick, all of which can be measured directly. Rearrangement of the above equation therefore gives the height of the tree as:

$$a' = \frac{ac'}{c}$$

Activity 6 Tree diagrams

Reread the last paragraph, starting from 'Provided that the day is sunny, ...', and identify any explicit or implicit modelling assumptions (including stressings and ignorings) that underlie the claim that the computational process described allows you to calculate accurately the height of a tree.

Figure 6 shows one common and important example of similar triangles.

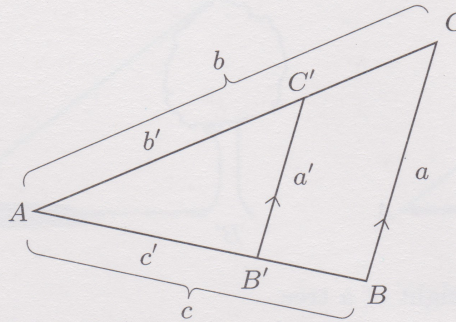


Figure 6 Similar triangles produced by parallels

Here one triangle lies inside the other. The two triangles share a vertex, at A; the side AB' of the triangle $AB'C'$ is part of the side AB of the triangle ABC , and the side AC' is part of the side AC ; the angle at B' of the triangle $AB'C'$ equals the angle at B of the triangle ABC , and the angle at C' equals the angle at C . The two triangles are similar because their angles are equal. The side $B'C'$ of triangle $AB'C'$ is parallel to the side BC of triangle ABC . Since the triangles are similar, you know that:

$$\frac{a'}{a} = \frac{b'}{b} = \frac{c'}{c}$$

2.2 Similarity, circles and measuring angles

The concept of similarity applies to plane figures other than ones made up of straight lines: in particular, it applies to circles. A scale drawing of any circle is another circle. Every circle is similar to any other. Thus, ratios of corresponding lengths are the same; and so in particular the ratio of the circumferences of any two circles is equal to the ratio of their diameters.

The circumference of a circle whose diameter is of unit length is denoted by π . (This is one way of specifying what the number π actually is.) Consider another circle, with circumference c and diameter d . The ratio of the circumferences is c/π , which is equal to the corresponding ratio of the diameters $d/1$. Hence:

$$\frac{c}{\pi} = \frac{d}{1}$$

from which the following formula for the circumference c of any circle with diameter d is found:

$$c = \pi d$$

This can be read as a statement that the circumference of a circle is directly proportional to its own diameter. Thus, the formula for the circumference of a circle records the fact that all circles are similar to each other. The number π is the constant of proportionality, and can also be thought of as the actual circumference of a circle with a diameter of 1 unit.

Note that, since $d = 2r$, where r is the circle's *radius*, the above formula can also be written as:

$$c = 2\pi r$$

The same idea as above crops up when considering the relationship between the length of an *arc* of a circle (a part of the circumference) and the angle at the centre that specifies it. Figure 7 shows an arc of a circle, bounded by two lines through the centre of the circle—two *radial* lines. The angle at the centre of the circle between the radial lines is called the *angle subtended by the arc*.

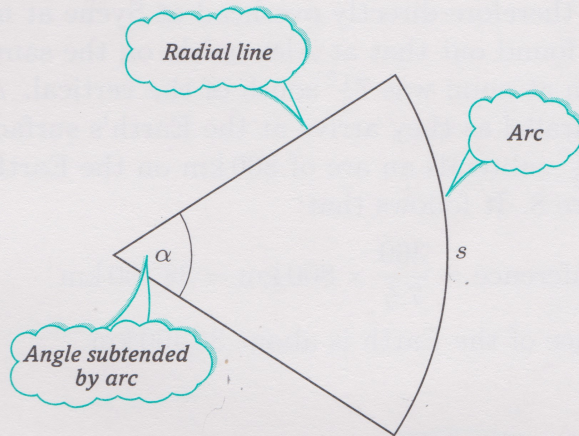


Figure 7

Clearly, the longer the arc, the larger the angle. The length of the arc is determined by the angle it subtends at the centre of the circle. In fact, the length of the arc is directly proportional to the angle it subtends. Suppose that the length of the arc is s and that the size of the corresponding angle is α° . Then

$$s = k\alpha$$

for some constant k .

► What is the value of k ?

The full circumference of the circle can be thought of as an arc that subtends an angle of 360° (a complete turn); and the length of the circumference is $2\pi r$, where r is the radius of the circle. So:

$$2\pi r = 360k$$

Rearranging gives:

$$k = \frac{2\pi}{360}r$$

Therefore:

$$s = k\alpha = \frac{2\pi}{360}r\alpha = \frac{\alpha}{360} \times 2\pi \times r \quad (1)$$

This shows that the length of an arc corresponding to a given angle α° is directly proportional to the radius of the circle of which it is part. This means that if each of two figures consists of two radial lines and an arc subtending the same angle at the centre, then the two figures are similar.

Recall that α is the Greek (lower-case) letter 'alpha'.

This formula for the length of an arc of a circle will be used in Section 4 to calculate the distance between any two places on the surface of the Earth.

Formula (1) can be rearranged as:

$$\frac{s}{2\pi r} = \frac{\alpha}{360}$$

So α is the same the proportion of 360 as s is of the circumference $2\pi r$. This fact was known to the Greeks around 250 BC, and was used by Eratosthenes to work out a value for the circumference of the Earth. It is said that one day when Eratosthenes was at a place called Syene in Egypt, which he knew to be some 800 km due south of Alexandria, he observed that the sun at midday shone directly down the shaft of a deep well, so that its light was reflected back directly into his eye as he looked into the well. The sun was therefore directly overhead at Syene at noon on that day. Eratosthenes found out that at Alexandria on the same day the direction of the sun at noon was $7\frac{1}{2}^\circ$ south of the vertical. Assuming the sun's rays to be parallel as they arrive at the Earth's surface, he deduced that an angle of $7\frac{1}{2}^\circ$ subtends an arc of 800 km on the Earth's surface, as illustrated in Figure 8. It follows that:

$$\text{Earth's circumference} = \frac{360}{7.5} \times 800 \text{ km} = 38\,400 \text{ km}$$

So the circumference of the Earth is about 38 400 km.

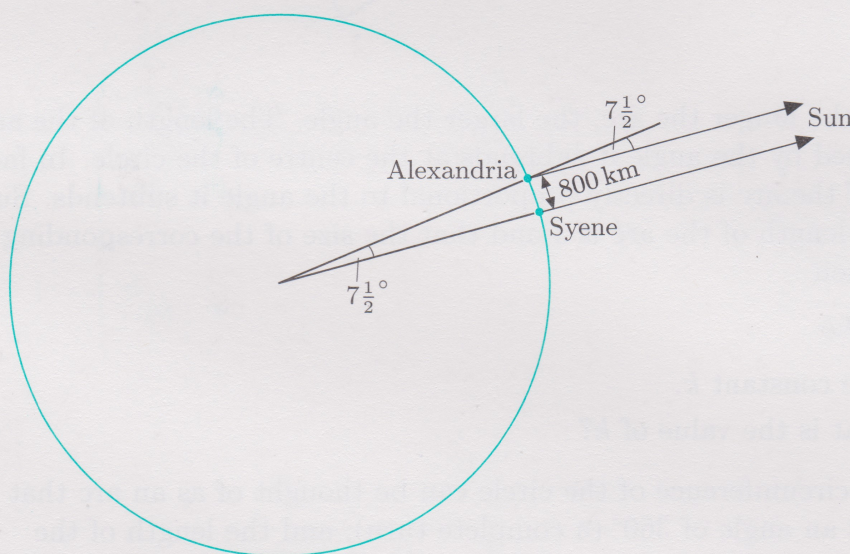


Figure 8

Activity 7 The Earth's radius

What is the radius of the Earth, according to Eratosthenes' calculations? What modelling assumptions can you identify underlying the calculation?

Formula (1), relating arc length to angle, may be read from right to left: so it also says that the angle is directly proportional to the length of the arc it subtends. This suggests the possibility of using arc length as a measure

Of course, Eratosthenes did not use kilometres but *stadest*: 1 stade $\simeq \frac{1}{10}$ mile. So Eratosthenes' figure of 5000 stades $\simeq$ 500 miles $\simeq$ 800 km.

of angle. The use of degrees to measure angles is a historical accident. Looked at from a mathematical point of view, it makes much more sense to think of the length of the arc as giving the measure of the angle.

Because arc length depends on the radius, in order to make this work it is necessary to fix the radius of the circle under consideration: the obvious choice is that it should be 1 unit. The resulting unit of measurement of angles is called the *radian*, which you first met in Chapter 9 of the *Calculator Book*. An angle measures s radians when the length of the arc of the circle of unit radius that it subtends is s units. For instance, if the radius is 1 m, and the arc length is also 1 m, then the angle is 1 radian.

The relationship between radians and degrees is that a complete turn is both 2π radians and 360 degrees; so 1 radian is $(360/2\pi)^\circ \simeq 57^\circ$, and 1° is $2\pi/360$ radians. There is no conventional shorthand notation for radians; however, if there are no units indicated for an angle (that is, if there is no degree sign $^\circ$) then, by convention, the angle measurement is in radians.

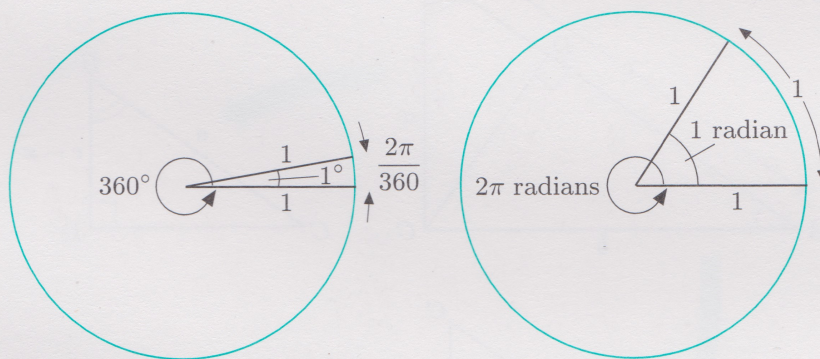


Figure 9 1° and 1 radian

Draftsmen and cartographers tend to use degrees; mathematicians, technologists and scientists tend to use radians. Talking about geographical as well as mathematical applications in this unit means using both degrees and radians.

- What is the equivalent formula to formula (1) when the angle α is measured in radians?

When α is a complete turn (2π radians), the equation $s = k\alpha$ gives $2\pi r = 2\pi k$, and so $k = r$. Therefore formula (1) becomes simply $s = r\alpha$.

So, the length s of an arc is the radius times the angle subtended, α , measured in radians. Rewriting $s = r\alpha$ as $\alpha = s/r$ tells us that α is the number of radii in the arc. In other words, measuring angles in *radians* is based on measuring arcs in *radii*.

Radii is the plural of *radius*.

In summary, the basic features of scale maps, that they represent distances to scale and angles exactly, underlie ideas of shape when talking about objects of different sizes having the same shape. These ideas are formalized mathematically in the concept of similarity, a relationship that may or may not hold between any two plane figures. Two plane figures are similar

when one is a scaled version of the other. Two triangles are similar when they have the same angles; this is a special property of triangles. But the idea of similarity is not restricted to figures with straight edges, and has important consequences when it is applied to circles and circular arcs, as you will see in Section 3.

2.3 Pythagoras' theorem revisited

Recall that you met and used this result in *Unit 6*, and again in Section 1 of this unit.

An example of the use of similar triangles comes in a proof of Pythagoras' theorem.

For two right-angled triangles to be similar, it is enough to know that just one of the angles other than the right angle in one of them is equal to an angle in the other. This general fact has been subtly used in Figure 10 to produce two similar right-angled triangles.

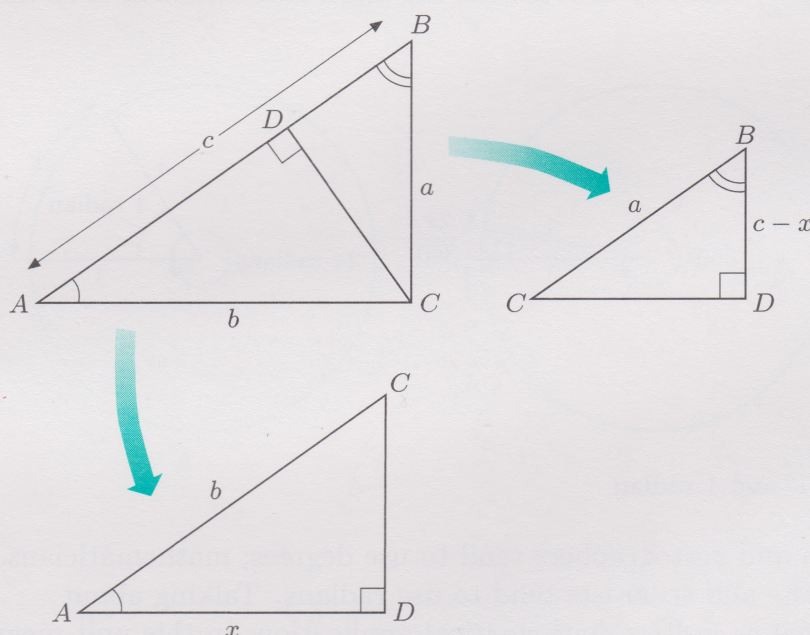


Figure 10

Here ABC is a right-angled triangle with the right angle at C . A line from C has been drawn perpendicular to the opposite side AB , which it meets at the point D . Thus, ACD is also a right-angled triangle, with the right angle at D ; and since the angle at A is common to both ACD and ABC , these two triangles are similar to each other. The corresponding pairs of sides are: CD and BC (the sides opposite A); AC and AB (the two hypotenuses); and the remaining pair, AD and AC .

Call the length of AD x . By the fact of the similarity of the two triangles:

$$\frac{x}{b} = \frac{b}{c} \quad (2)$$

Now triangle BCD is also a right-angled triangle, which shares the angle at B with ABC . Thus, BCD is similar to ABC , with corresponding pairs of sides: CD and AC ; BC and AB ; BD and BC .

The length of BD is $c - x$, since the length of AB is c and the length of AD is x . Using the ratio property again gives:

$$\frac{c - x}{a} = \frac{a}{c} \quad (3)$$

From equation (2):

$$x = \frac{b^2}{c} \quad (4)$$

From equation (3):

$$c - x = \frac{a^2}{c} \quad (5)$$

Substituting the expression for x from equation (4) into equation (5) gives:

$$c - \frac{b^2}{c} = \frac{a^2}{c}$$

Therefore:

$$c = \frac{a^2}{c} + \frac{b^2}{c} = \frac{a^2 + b^2}{c}$$

So finally:

$$c^2 = a^2 + b^2$$

This is the relationship that constitutes Pythagoras' theorem.

Activity 8 Why is it true for all?

Look back over the above argument and identify any points you have difficulty following. Can you explain *why*, on the basis of the above argument, Pythagoras' theorem has to be true for *any* right-angled triangle?

2.4 Areas of plane figures

Unit 6 made brief mention of areas, and explored the relationship between areas on maps and actual areas on the ground. The general result (for a 1:25 000 scale map) was found to be:

$$\text{area on the ground} = (25\,000)^2 \times \text{area on the map}$$

This subsection will explore some of the mathematical relationships between scaling and similarity, focusing on areas of plane figures. The main general question for this subsection is the following.

- How can the area of any plane figure with straight sides be worked out?

The reading associated with *Unit 13* entitled 'Sorry, no King Kongs' made use of various results about scaling areas and volumes.

There is a commonly available puzzle of ancient Chinese origin known as a *tangram*. Often made of wood, it consists of seven pieces of particular sizes and shapes that together fit together exactly to make up a square, as shown in Figure 11.

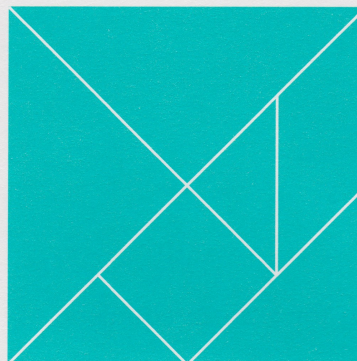


Figure 11

Challenges are to rearrange these pieces to make particular other shapes, whose outlines only are given. Two examples are shown in Figure 12.

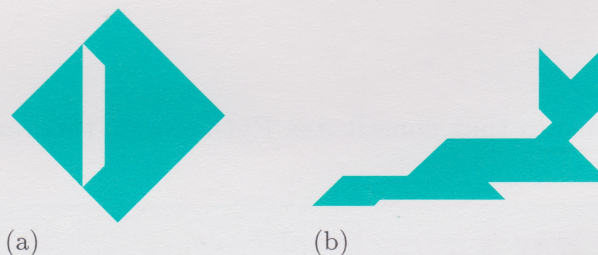


Figure 12

The Chinese tangram

To a certain extent the Chinese tangram resembles the Western jigsaw puzzle, but it differs from the jigsaw in always having the same number of pieces, which are fitted together in different ways to make a large number of different shapes. The seven pieces of the tangram are formed from a square, as shown in [Figure 11] (without the diagram it is quite difficult to make the square from the pieces). The idea is to use the pieces to construct certain figures. These may be geometrical—such as a triangle, trapezium or parallelogram—or representational—human figures running, sitting, falling, playing, dancing, fish swimming, cats or pigs lying down, bridges, shops, houses, etc.

(Joost Elffers (1976)

Tangram: the ancient Chinese shapes game, Penguin, p. 7)

From the point of view of mathematics, one thing this puzzle illustrates is that quite different shapes can have exactly the same area. Whatever else is true, each varied configuration that can be made with all of the pieces laid flat with no overlaps must have precisely the same area as the square

(and the square is the basic reference and comparison figure for area computations in Western culture, which is why the units are called *square* metres, *square* inches, and so on).

This puzzle provides the initial strategic idea for ways of working out areas of plane figures with straight sides (called *rectilinear* figures): cut the figure up into certain cleverly—and conveniently—chosen shapes, so the total area is conserved, and then rearrange them into a shape whose area is easier to work out.

To provide an example, consider the triangle in Figure 13.

► How might you calculate its area?

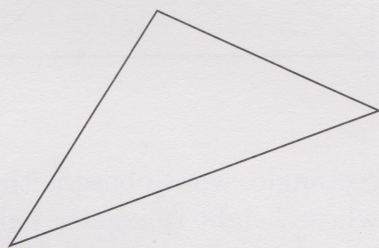


Figure 13

You may recall a formula for the area of any triangle as: ‘half the base times the height’.

► But where does such a formula come from? How can you see that it is right?

Here is a procedure described in words. It provides another example of an *algorithm*, a term introduced in *Unit 13*. It purportedly tells you what to do, no matter which particular triangle you have in front of you. So one thing to be thinking about is: how can I be sure that I can always do what it is telling me to do, no matter what triangle I have? And secondly: will any of these transformations alter the area?

- (a) First, rotate the triangle so its longest side is horizontal and the triangle lies above this horizontal *base*, as in Figure 14.
- (b) Next, draw in the vertical line from the top vertex to the horizontal base. (This line, because it meets the base at a right angle, is called a *perpendicular* to the base, and the action of drawing such a line is frequently referred to as *dropping a perpendicular*.) Then draw in a horizontal line parallel to the base that divides the height to the top vertex exactly in half. Both lines are shown in Figure 15.

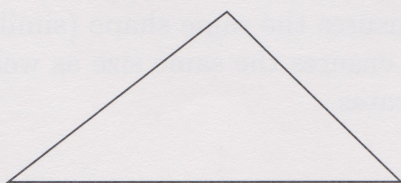


Figure 14

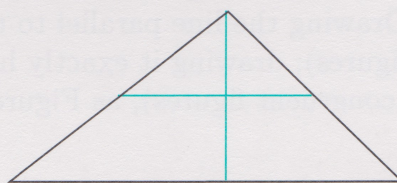


Figure 15

- (c) Next, rotate the two smaller triangles that have been created above the halfway horizontal, the left one to the left and the right one to the right, as shown in Figure 16. Notice that they fit together exactly with the bottom part of the triangle.

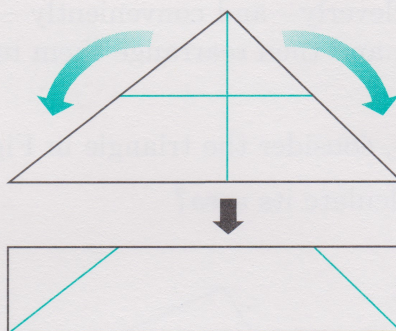


Figure 16

What you now have is a rectangle, whose base is the same as the base of the original triangle and whose height is exactly half the height of the original triangle. And this rectangle has exactly the same area as the original triangle. The area of a rectangle is base times height, and so the area of this rectangle—and hence that of the original triangle—is base times half the vertical height of the triangle.

So this way of looking at the starting figure, as a tangram, produces a formula that reflects this way of ‘seeing’ the situation:

$$\text{area of triangle} = \text{base} \times \text{one-half vertical height} \quad (6)$$

In your mind’s eye, you may be able to see that every step will always be possible to achieve.

- ◇ Any triangle must have a longest side (if there is more than one longest, as with an isosceles or equilateral triangle, just pick one of them), and merely rotating a triangle does not affect its area.
- ◇ The construction lines can always be drawn, and so the smaller triangles, above the halfway horizontal, can always be formed.
- ◇ The hardest thing to be sure of is that the two smaller triangles will always fit exactly with the bottom part of the triangle to produce a rectangle.

► Why must this be so?

The secret is in matching up congruent figures: two plane figures (in this case, triangles) are *congruent* if they are the same shape and size. Drawing the line parallel to the base ensures the same shape (similar figures); drawing it exactly halfway up ensures the same size as well (congruent figures), as Figure 17 illustrates.

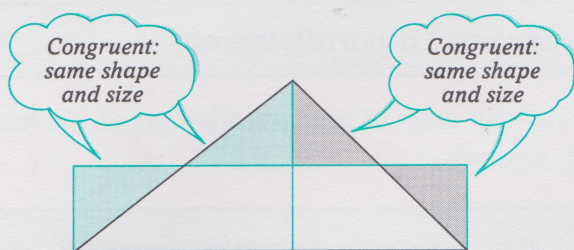


Figure 17

Underlying this particular justification for a formula for finding the area of any triangle is a way of dealing with the areas of more irregular and complex plane figures. It is relatively straightforward. If you can find a way to divide up a given plane figure into smaller shapes and then rearrange them in such a way (with no overlaps and no bits left out) as to give a second plane figure whose area you know how to find, then the area of the two figures will be the same.

Activity 9 Area of an equilateral triangle

In *Unit 12*, after the discussion of the snowflake curve in Example 5, the area of an equilateral triangle was claimed to be $\frac{1}{4}\sqrt{3}l^2$, where l is the side length of the triangle. Use Pythagoras' theorem and formula (6) to justify this assertion.

Example 2 Finding the area of an irregular pentagon

What is the area of the irregular pentagon in Figure 18?

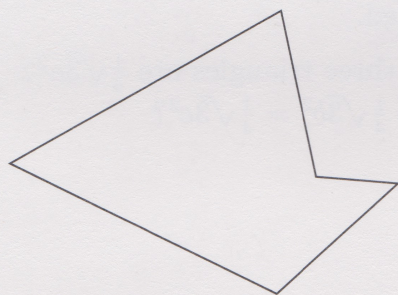


Figure 18

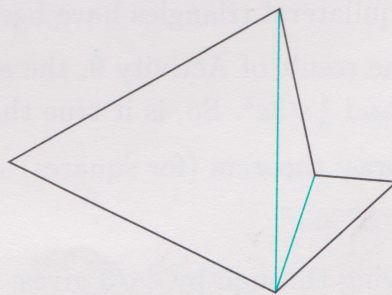


Figure 19

One strategy is to cut the figure into triangles and then find the area of each triangle using formula (6). One way to cut the figure into triangles is shown in Figure 19.

A *parallelogram* is a quadrilateral with opposite sides parallel.

Activity 10 Area of a parallelogram

What is a formula for the area of any parallelogram, in terms of its base and vertical height, and why?

Activity 11 More triangle formulas (harder)

The formula for the area of a triangle can be written in the following three equivalent ways:

- (a) one-half (base $\times$ vertical height);
- (b) base $\times$ (one-half vertical height);
- (c) (one-half base) $\times$ vertical height.

The unit text above provided a way of seeing that the second formulation is correct. Can you find a way of seeing that each of the other two equivalent formulations is *geometrically* correct?

Activity 12 Every picture tells a story

Look back at the way geometric diagrams have been used in this subsection. What strikes you about the role they have been made to play?

Example 3 More on Pythagoras' theorem

Pythagoras' theorem is stated for squares. Is it true for equilateral triangles? In other words, is it true that, for any right-angled triangle, the equilateral triangle on the hypotenuse is equal to the sum of the equilateral triangles on the other two sides?

Figure 20 shows a right-angled triangle with sides of length a , b and c on which equilateral triangles have been constructed.

Using the result of Activity 9, the areas of the three triangles are $\frac{1}{4}\sqrt{3}a^2$, $\frac{1}{4}\sqrt{3}b^2$, and $\frac{1}{4}\sqrt{3}c^2$. So, is it true that $\frac{1}{4}\sqrt{3}a^2 + \frac{1}{4}\sqrt{3}b^2 = \frac{1}{4}\sqrt{3}c^2$?

Pythagoras' theorem (for squares) says that:

$$a^2 + b^2 = c^2$$

Multiplying through by $\frac{1}{4}\sqrt{3}$ gives:

$$\frac{1}{4}\sqrt{3}a^2 + \frac{1}{4}\sqrt{3}b^2 = \frac{1}{4}\sqrt{3}c^2$$

So the result holds for equilateral triangles too.

The reason the result holds for equilateral triangles as well as for squares is that the area of an equilateral triangle is directly proportional to the area of the square with the same side length, with constant of proportionality $\frac{1}{4}\sqrt{3}$.

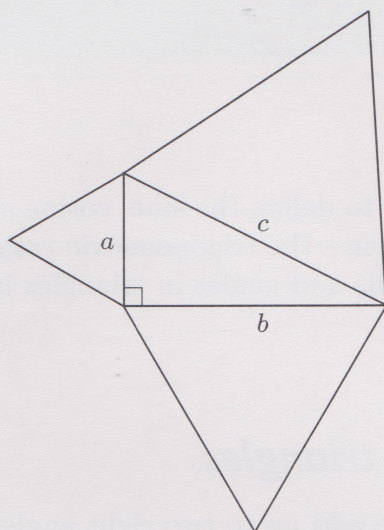


Figure 20

Activity 13 Semicircular Pythagoras

Is it true that, for any right-angled triangle, the semicircle on the hypotenuse is equal to the sum of the semicircles on the other two sides?

In fact, Pythagoras' theorem holds for any plane figure whose area is directly proportional to the square of some linear measurement of that figure. So, for example, it holds not only for squares, equilateral triangles and semicircles but also for circles and all regular polygons.

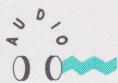
The area of a regular polygon with n sides is discussed in Subsection 3.3.

Outcomes

After studying this section, you should be able to:

- ◇ explain what is meant by the word 'similar' when it is applied to pairs of plane figures, and recognize whether or not two figures are similar to each other (Activity 5);
- ◇ explain how to use the properties of similar figures to solve simple geometrical problems, such as the calculation of the height of a tree from the length of its shadow (Activity 6);
- ◇ explain the role of similarity in the definition of π , in the definition of the radian measure of angles and in the calculation of the lengths of arcs of circles (Activity 7);
- ◇ identify modelling assumptions underlying calculations (Activities 6, 7);
- ◇ follow a proof of Pythagoras' theorem based on the properties of similar triangles, and explain why the theorem and modifications of it hold (Activities 8, 13);
- ◇ calculate areas of various plane figures (Activities 9, 10, 11);
- ◇ make some observations about the nature and use of geometric diagrams in mathematical argument (Activity 12).

3 Trigonometry in the plane



Aims This section aims to define the sine, cosine and tangent of an angle, and then to use them—the trigonometric ratios, as they are called—to calculate lengths and angles in triangles in the plane. ◇

3.1 Right-angled triangles

The angles of any triangle add up to two right angles ($= 180^\circ = \pi$ radians). The two non-right angles of a right-angled triangle together add up to one right angle. Thus, if two right-angled triangles have one non-right angle equal, they must be similar to each other. The ratio of any pair of corresponding sides is therefore the same for all right-angled triangles containing that angle, and so depends only on the angle.

This is the basis for a useful method for calculating lengths and angles in right-angled triangles, which can be extended to many other plane figures. It depends on the scaling property of similar triangles being arranged in a different way from before. Recall that the scaling property is given by:

$$a' = ka, \quad b' = kb, \quad c' = kc$$

From this it follows that:

$$\frac{a'}{b'} = \frac{a}{b} \quad \frac{a'}{c'} = \frac{a}{c} \quad \frac{b'}{c'} = \frac{b}{c}$$

These formulas express the fact that the ratio of the lengths of two sides from one triangle is equal to the ratio of the lengths of the corresponding sides from any similar triangle. In other words, the *ratio* of a pair of sides is the same for all similar triangles. It is *invariant*. This differs from the previous formulation (on page 16), which was concerned with the ratios of pairs of corresponding sides from *different* similar triangles. Trigonometric ratios arise from using this new formulation of the scaling property of similar triangles in the special case when the triangles are right-angled.

To take a simple example, suppose that you know all the angles of a right-angled triangle, and you also know the length of one of its sides. Such an example is shown in Figure 21. Suppose further that someone had conveniently tabulated the values of the ratios of the various pairs of sides for a range of angles (as happened historically). Then all you would have to do would be to look up in the table the value of the appropriate ratio for the known value of the angle, and multiply by the length of the side that you did know. For example, if, for an angle of 30° at A in Figure 21, the tabulated value of a/c is 0.5, then the length of side BC is:

$$a = \frac{a}{c} \times c = 0.5 \times 1 = 0.5$$

The principle in operation here is that the values of the ratios, once tabulated, become a reusable resource for doing all sorts of calculations of lengths and angles.

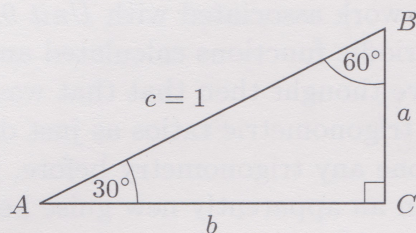


Figure 21

- If, for the angle of 30° at A in Figure 21, the tabulated value of b/c is 0.866, what is the length of side AC ?

Side AC has length:

$$b = \frac{b}{c} \times c = 0.866 \times 1 = 0.866$$

There are six different possible ratios of pairs of sides of a right-angled triangle, each of which has its special name, relative to the angle concerned. In Figure 22, the sides have been named relative to the angle at A .

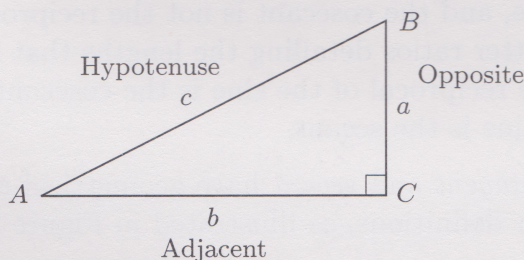


Figure 22 Sides of a right-angled triangle

The six possible ratios have the following names and standard abbreviations (in brackets):

sine A (written $\sin A$)	$= \frac{\text{opposite}}{\text{hypotenuse}}$	$= \frac{a}{c}$
cosine A (written $\cos A$)	$= \frac{\text{adjacent}}{\text{hypotenuse}}$	$= \frac{b}{c}$
tangent A (written $\tan A$)	$= \frac{\text{opposite}}{\text{adjacent}}$	$= \frac{a}{b}$
cotangent A (written $\cot A$)	$= \frac{\text{adjacent}}{\text{opposite}}$	$= \frac{b}{a}$
secant A (written $\sec A$)	$= \frac{\text{hypotenuse}}{\text{adjacent}}$	$= \frac{c}{b}$
cosecant A (written $\operatorname{cosec} A$)	$= \frac{\text{hypotenuse}}{\text{opposite}}$	$= \frac{c}{a}$

A trigon is an alternative name for a triangle: 'tri' means 'three' and 'gon' comes from the Greek for 'angle'. So the word 'trigono-metry' can be seen to refer to the measurement of triangles, in the same way that 'geo-metry' alludes to the measurement of the Earth (*ge* is Greek for the Earth), something this unit turns to in the next section.

Remember that the reciprocal of x is $1/x = x^{-1}$.

These ratios are called *trigonometric ratios*, and the study of the trigonometric ratios is called *trigonometry*.

You met the terms *sin*, *cos* and *tan* in an apparently very different context in your *Calculator Book* work associated with *Unit 9*, where they were used as the names of periodic functions calculated and graphed by your calculator. You may have thought then that that was odd, if you had previously met them as trigonometric ratios as just defined. On the other hand, if you have not done any trigonometry before, you may be puzzled to find the same terms in an apparently new guise here. One way or another, you have two sets of apparently unrelated concepts with the same names. In the next unit, you will see how the two versions of *sin*, *cos* and *tan* are actually different aspects of the same thing.

The names of the trigonometric ratios have the arbitrary appearance that comes from them having grown rather than having been designed. But there is some logic to them.

First, they come in pairs: each of sine, tangent and secant has its accompanying 'co-' version. The prefix 'co-' is associated with taking the *complementary* angle (which is the other non-right angle in the triangle): thus, the *cotangent* is the *tangent* of the *complementary* angle, and so on.

There is an additional complication: although the cotangent is the reciprocal of the tangent (that is, $\cot A = 1/\tan A$), the cosine is not the reciprocal of the sine, and the cosecant is not the reciprocal of the secant. If you look at the letter ratios detailing the lengths that specify each one, you will see that the reciprocal of the sine is the cosecant and the reciprocal of the cosine is the secant.

Second, the words *tangent* and *secant* have geometrical meanings associated with their definitions, as illustrated in Figure 23.

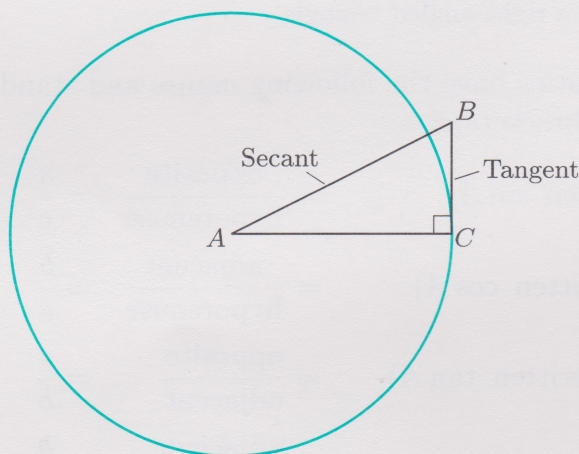


Figure 23 Tangent and secant

Consider the right-angled triangle ABC . For a circle with centre at A and radius equal to the length of the adjacent side (with respect to the angle at A), the opposite side is *tangent* to the circle, because it touches it; and the hypotenuse is the only line of the triangle that intersects the circle, and is therefore called the *secant*. If the figure is scaled so that the circle has unit radius, then the tangent of A ($\tan A$) is simply the length of the tangent line BC in the figure, and the secant of A ($\sec A$) is simply the length of the secant line AB in the figure.

The Latin verbs *tangere* and *secare* respectively mean 'to touch' and 'to cut' (as in 'secateurs' or 'intersect').

It is unfortunate that there is no reasonable explanation of the origin of the term *sine*. Apparently, it is due to a mistranslation from an Arabic text in the Middle Ages.

It is not really necessary to have names for all six of the ratios—the fact that there are is really a historical quirk. It is usually enough to know any two of them (though not two that are reciprocals of each other, such as tangent and cotangent): the rest can then be expressed in terms of the basic two. The basic ratios are usually taken to be sine and cosine. The others can then be expressed as follows:

$$\tan A = \frac{\sin A}{\cos A}$$

$$\cot A = \frac{1}{\tan A} = \frac{\cos A}{\sin A}$$

$$\sec A = \frac{1}{\cos A}$$

$$\operatorname{cosec} A = \frac{1}{\sin A}$$

Activity 14 Handbook activity

Look back at the notes you made about the trigonometric ratios when you were studying *Unit 9* and the associated chapter of the *Calculator Book*, and now add to them to extend your Handbook entry on trigonometric ratios.

The trigonometric ratios may appear to be rather abstract; and it may not be obvious why they could be useful. However, if you think back to *Unit 6*, you may realize that you have come across two of these ratios before, in a context that is certainly of practical importance: measuring the steepness of the slope of a road up a hill. Figure 24 (overleaf) shows a simplified model of the hill as a right-angled triangle.

This model of a hill was also used in *Unit 6*.

It would be natural to take the size of the angle at A as the measure of the steepness of the hill: clearly, the larger the angle, the steeper the hill. But as you will recall, in neither of the two ways in which slopes are actually measured is the angle mentioned explicitly. The two alternative measures of the slope, or gradient, of the road both involve the ratio of a pair of

sides of the triangle. The first, called the *mathematical gradient* in *Unit 6*, is the ratio of the change in height to the horizontal distance. In terms of the symbols in Figure 24:

$$\text{mathematical gradient} = \frac{a}{b} = \frac{\text{opposite}}{\text{adjacent}} = \tan A$$

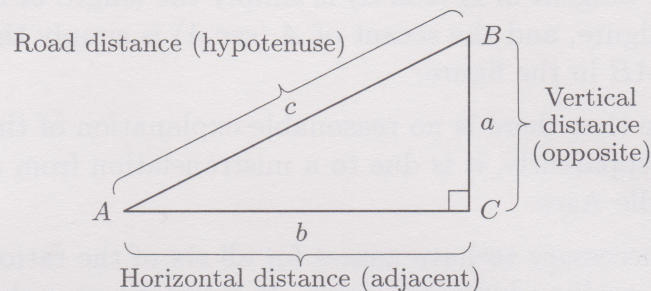


Figure 24 Triangle model of a hill (names relative to angle at A)

The *road gradient*, on the other hand, is worked out by dividing the change in height of the road between two points by the distance between them measured along the surface of the road. The road gradient is the ratio of the change in height to the road distance:

$$\text{road gradient} = \frac{a}{c} = \frac{\text{opposite}}{\text{hypotenuse}} = \sin A$$

It was implicit in the description of gradients in *Unit 6* that, provided you keep to a *single* triangle model of the hill, it is unimportant along which section of the road you choose to take the measurements needed to calculate the gradient. From the vantage point of the present unit, it is now clear that this was a covert appeal to the properties of similar triangles.

From the above formulas, it follows that the gradient of a hill whose side makes a known angle with the horizontal can be found directly by using a calculator to evaluate the tangent (in the case of the mathematical gradient) or sine (in the case of the road gradient). A hill that rises at an angle of 10° has a (mathematical) gradient of $\tan 10^\circ \simeq 0.18$: that is, about 18%. Its road gradient is $\sin 10^\circ \simeq 0.17$, or 17%.

For a comparatively small angle, the difference between the mathematical and road gradients is small; but for larger angles it becomes significant. At an angle of 30° , the mathematical gradient is about 58% ($\tan 30^\circ \simeq 0.58$), while the road gradient is exactly 50% ($\sin 30^\circ = 0.5$).

To find the angle of slope corresponding to a gradient given in the usual way, as 1 in 10, or 10%, it is necessary to work backwards: *from* knowing the value of the tangent of an angle (or its sine, as the case may be), *to* find the size of the angle itself. This requires the use of the *inverse function*. For each of the trigonometric ratios, thought of as trigonometric *functions*, the inverse function (the one that undoes the effect of the original function) is traditionally written with a small $^{-1}$ raised like an exponent, such as $\tan^{-1}$ or $\sin^{-1}$. Your calculator has built-in ways of computing these inverse functions. It gives $\tan^{-1} 0.1 \simeq 5.7^\circ$, and so a road whose mathematical gradient is 1 in 10 makes an angle of just under 6° with the horizontal.

Inverse and reciprocal

You have already encountered the notation $^{-1}$ to mean the reciprocal:

$$2^{-1} = \frac{1}{2} \quad \left(\frac{1}{2}\right)^{-1} = 2 \quad a^{-1} = \frac{1}{a}$$

Thus, a likely interpretation (but unfortunately an incorrect one) of $\tan^{-1}(10^\circ)$ is $\frac{1}{\tan(10^\circ)} = \cot(10^\circ)$. But the cotangent is quite a different function from the inverse tangent.

For a variety of reasons, $\tan^{-1}$ is one common (potentially confusing) notation for the inverse function to $\tan$. Another way of writing this function is as ‘arctan’ (that is, the angle—measured by the *arc* length when using radians—whose *tangent* is ...). Neither of these notations is perfect: you are likely to come across both of them in the course of your mathematical studies.

Activity 15 Road gradients

- Use your calculator to find the gradients (mathematical and road) of a hill that makes an angle of 15° with the horizontal.
- Use your calculator to find the angle a road makes with the horizontal if its mathematical gradient is 15%? What if its *road* gradient is 15%?
- Compile an entry on road gradients for your Handbook.

The remainder of this subsection provides some practice on using the trigonometric ratios. To begin, consider a particularly simple triangle; in fact, the simplest right-angled triangle the lengths of whose sides are whole numbers (of whatever units of length are in use), as shown in Figure 25. This famous triangle, which is the simplest application of Pythagoras’ theorem ($3^2 + 4^2 = 5^2$), crops up frequently (especially in problems involving right-angled triangles where the problem-setter wants the triangle to have nice simple dimensions).

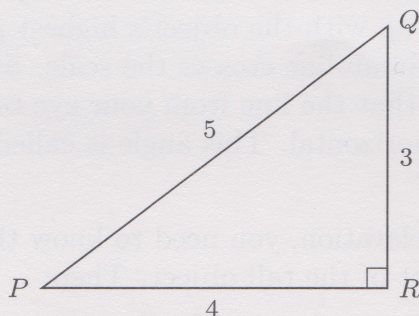


Figure 25 A (3, 4, 5) triangle

Activity 16 The (3, 4, 5) triangle

- For the angle at P in Figure 25, write down the value of each of the following trigonometric ratios: $\sin P$, $\cos P$, $\tan P$, $\operatorname{cosec} P$, $\sec P$, $\cot P$. (Leave your answers as fractions—there is no need to use your calculator!)
- For the angle at Q , write down the value of each of the following trigonometric ratios (written in a different order): $\cos Q$, $\operatorname{cosec} Q$, $\cot Q$, $\sec Q$, $\sin Q$, $\tan Q$.
- Use your calculator to find the sizes of the angles at P and Q , in degrees.

In Section 1, you saw how to work out the height of a tall vertical object, such as a tree, by using similar triangles. Here is another way of doing it, using trigonometry. It has the advantage that you do not have to wait until the sun comes out. Moreover, it can be used for objects that fail to cast shadows and when the ground is not flat.

In order to use the trigonometric method, you have to be able to measure angles. Figure 26 shows a device for doing so, sometimes called a *clinometer* (because it measures *inclines*), which is made by modifying a protractor.

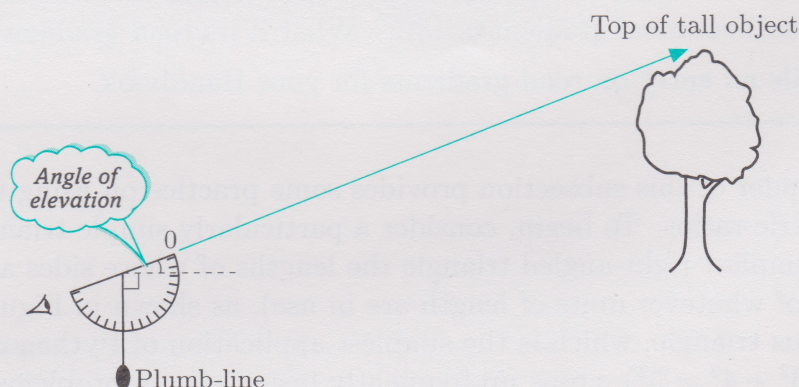


Figure 26 Modifying a protractor to measure angles of elevation

Point the zero of the protractor towards the tall object, squint along the straight side and line it up with the object's highest point. Make a note of the reading where the plumb-line crosses the scale. Subtract 90° from this, and you have the angle that the line from your eye to the top of the tall object makes with the horizontal. This angle is called the *angle of elevation* of the object.

As well as the angle of elevation, you need to know the horizontal distance from your eye to the foot of the tall object. Then:

$$\tan (\text{angle of elevation}) = \frac{\text{height of tall object}}{\text{distance to its foot}}$$

From this it follows that:

$$\text{height of tall object} = \text{distance to its foot} \times \tan(\text{angle of elevation})$$

You need to be a bit careful though, because the height given by this formula is from the point (eye level) where the angle of elevation is measured, as the following example illustrates.

Example 4 How high is the cliff?

Umberto, who is 1.8 m tall, stands 72 metres from the base of a vertical cliff and finds that the angle of elevation of the top of the cliff is 67° , as illustrated in Figure 27. How high is the cliff?

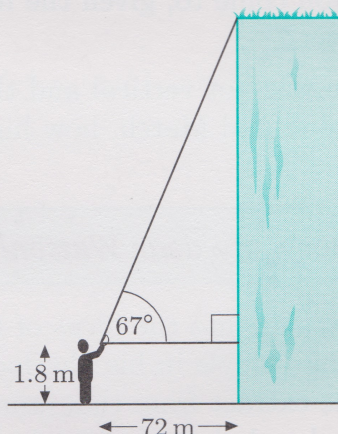


Figure 27

$$72 \times \tan 67^\circ \simeq 72 \times 2.36 \simeq 169.6$$

So the top of the cliff is about 169.6 metres above Umberto's head, and so the cliff is about $169.6 + 1.8 \simeq 171$ metres high.

It is often best, when measuring angles of elevation, to try to ensure that eye level is level with the base of the object being measured.

Activity 17 Getting from A to B

Suppose you are at point A and want to get to point B . You have located both points on the map; and you see that a certain point—point C —is 6.3 km due north of point A and 4.1 km due west of point B . What is the map bearing of point B from point A ? Use the answer to work out how far point B is from point A . Check your result by using Pythagoras' theorem.

Activity 18 *Finding the height of a tree (harder)*

- (a) On 21 March, one year, in Manchester, the angle of elevation of the sun at its highest point (called its *zenith*) is 36.5° , measured from ground level. At this time, the same year, also in Manchester, a tree had a shadow 30 metres long, measured from the centre of its trunk. Given that the ground was horizontal and that the tree rose to a point, for instance as a larch does, how high was the tree?
- (b) The previous 21 December, in Manchester, the angle of elevation of the sun at its zenith was 13° , again measured from ground level. At that time, the same tree had a shadow 94 metres long, measured from the centre of its trunk. How high does this make the tree at that time? What conclusion can you come to, given the likely temperatures in Manchester in the winter?
- (c) Assuming that the tree was not vertical and that its height did not change between December and March, how high was the tree?

**Activity 19** *Elevatory, my dear Watson!*

An interesting literary excursion into the ideas of this subsection is provided by Sir Arthur Conan Doyle in his Sherlock Holmes story 'The Musgrave Ritual'. Read the extract from that story in the reader now, and note down the mathematical techniques referred to in the extract.

3.2 General triangles

You have seen how trigonometric methods can be used to find unknown angles and side lengths in right-angled triangles. It is also possible to use such methods to find unknown angles and side lengths when the triangles are not right-angled. One important formula in this respect is the *cosine formula*. For a triangle labelled as in Figure 28, the cosine formula for the angle at C is given by:

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab} \quad (7)$$

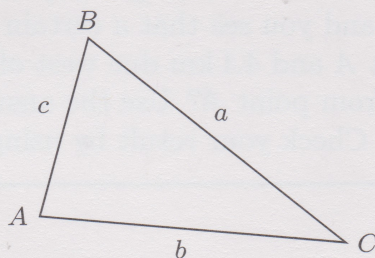


Figure 28

A proof of the validity of formula (7) is given in Section 1 of the Appendix.

In fact, the cosine formula can be applied to *any* of the angles of a triangle when the lengths of its sides are known. The expression on the right-hand side can be written in words in such a way as to make it clear how to apply the formula to any one of the angles in the triangle:

the sum of the squares of the lengths of the two adjacent sides minus the square of the length of the opposite side, all divided by twice the product of the lengths of the two adjacent sides.

Activity 20 *From other points of view*

Write down the equivalent symbolic versions of formula (7) for $\cos B$ and $\cos A$.

The cosine formula can be used to find the length of one side of a triangle when the other two sides and the opposite angle are known. For this purpose, it is best written in the equivalent form:

$$c^2 = a^2 + b^2 - 2ab \cos C \quad (8)$$

Writing it this way round shows that the cosine formula is similar in form to Pythagoras' theorem, but has an extra term involving $\cos C$. Of course, if the angle C happens to be a right angle ($= 90^\circ$), then $\cos C$ is zero, and the cosine formula reduces precisely to Pythagoras' theorem—as it should!

This is an example of a kind of test that can often be carried out on a new mathematical result: it confirms that the new result is consistent with what has gone before, and shows indeed that the new result is a generalization of the old result, which becomes, in turn, a special case of the new result. You can think of the final term $-2ab \cos C$ as a 'correction factor' applied to compensate for the fact that the triangle is not right-angled. When it is a right-angled triangle, the correction factor is zero.

Here is an example of using the cosine formula to find angles.

Example 5 *Garden planning with trigonometric tools*

Suppose that you are thinking of making some improvements to your garden. In order to do so, you want to draw an accurate scale plan of it, showing the positions of various trees and shrubs that you do not want to move. Measuring angles in the garden is not very easy; measuring distances, on the other hand, is quite straightforward (with the aid of a tape measure). So you have measured the distances between those objects in the garden that you want to put on your plan. When it comes to drawing the plan, however, it would be helpful to know the angles.

Figure 29 shows a representation of the various distances you have measured—but this picture is by no means reliable as a scale plan of the garden.

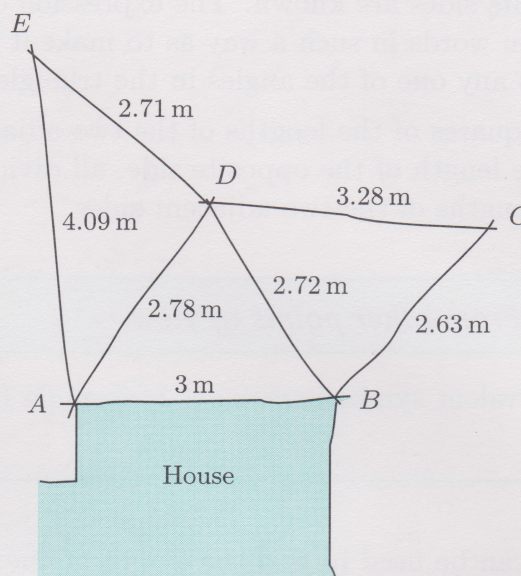


Figure 29 Rough sketch plan of the garden

To find the angle at A in triangle ABD , for example, you can use the cosine formula:

$$\cos A = \frac{2.78^2 + 3^2 - 2.72^2}{2 \times 2.78 \times 3} = \frac{9.33}{16.68} = 0.559\,352\,518 \text{ (to 9 decimal places)}$$

Therefore the angle at A is:

$$\cos^{-1} 0.559\,352\,518 = 55.988\,968\,13 \simeq 56^\circ$$

(It is pointless here to give the final answer to more significant figures than you could hope to measure with a protractor when drawing your scale plan; but intermediate steps should be carried out to the full accuracy of the calculator, to avoid possible rounding errors in the final answer.)

Activity 21 Using the cosine formula

Find the angle at B in triangle ABD in Figure 29; and also the angle at D .

The cosine formula is rather tedious to use if you have to keep applying it over and over again. But it can easily be programmed for the calculator. To save time, you can use the cosine formula program, details of which you will find in the *Calculator Book*.

Read Section 14.1 of Chapter 14 of the Calculator Book for details of the cosine formula program.



Activity 22 *Completing the garden plan*

Having completed Activity 21, you are in a position to locate point D on the garden plan, and to draw in the lines AD and BD (assuming that you have already drawn AB , the side of the house, which serves as a baseline). But to complete your plan you need to find the angles of triangles ADE and BCD . Find these angles and hence complete the plan of your garden.

There is another well-known formula that relates the side lengths and angles of any triangle, known as the *sine formula*. For a triangle labelled as in Figure 28, it is:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} \quad (9)$$

A proof of the validity of formula (9) is given in Section 2 of the Appendix.

The sine formula is very useful for surveying and map-making, where one side length and two angles of a triangle are known and the lengths of the other sides are required. Using this formula for surveying, navigation and so on is known as *triangulation*.

Activity 23 *Triangulation*

A surveyor chooses two points, A and B , on flat land exactly 1 km apart and then takes the bearings of a number of other locations from each of A and B . One of these locations is a church, C . B is due north of A . The bearing of C from A is 45° and that of C from B is 120° . Use the sine formula to calculate the distance of C from both A and B , to the nearest metre.

Activity 24 *Pick of the week*

The audiotape band for this unit contains extracts from broadcast radio illustrating the use of trigonometric, geometric and other mathematical concepts in a variety of contexts. As you listen, jot down instances of the use of such concepts.

Now listen to band 2 of Audiotape 4.



3.3 Using trigonometry to calculate π

It was pointed out in Subsection 2.2 that π is a constant of proportionality between the circumference of a circle and its diameter—but no mention was made of how the value of this constant could be found.

Approximations to the value of π were made, by Archimedes, in the third century BC. His idea was to approximate the circumference of the unit circle (that is, a circle of radius 1 unit) by the perimeter of a regular polygon inscribed in it (that is, drawn inside it and touching it at the vertices).

A *regular* polygon is one in which all the sides are of the same length and all the angles are the same. So, for instance, a square is a regular 4-gon and an equilateral triangle is a regular 3-gon.

Suppose you divide the 360° angle at the centre of the circle into a number of equal parts, draw the corresponding radial lines, and then join together the points of intersection of these lines with the circle, as shown in Figure 30. Then the perimeter of the regular polygon so constructed is an approximation to the circumference of the circle itself. The larger the number of equal parts into which the angle at the centre of the circle is divided (that is, the larger the number of sides of the inscribed regular polygon), the better the approximation.

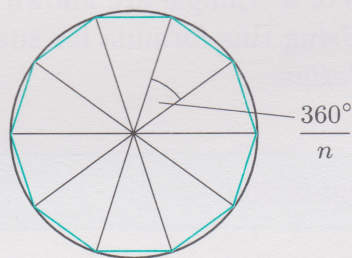


Figure 30 A regular polygon inscribed in a circle

Suppose that the polygon has n sides. Because it is a *regular* polygon, the angle at the centre of each of the triangles making up the polygon is therefore $(\frac{360}{n})^\circ$. To calculate the length of the opposite side of any one of these triangles, first draw a perpendicular from the centre to the opposite side, as in Figure 31. The triangle has two equal sides (the sides that are radii of the circle); the perpendicular line you have drawn therefore divides it into two identical halves. The angle at the centre of the right-angled triangle OPQ in the figure is $(\frac{180}{n})^\circ$; the length of OP , which is a radius of the unit circle, is 1; so the length of PQ is just $\sin(\frac{180}{n})^\circ$. Therefore the full length of every side of the polygon is twice this: $2 \sin(\frac{180}{n})^\circ$.

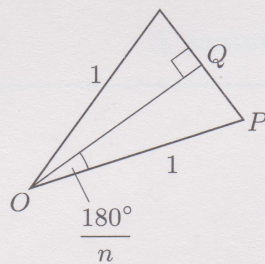


Figure 31

As the polygon has n such sides, its perimeter is therefore $2n \sin(\frac{180}{n})^\circ$. This should be approximately the circumference of the circle, which is exactly 2π (since $r = 1$). So $n \sin(\frac{180}{n})^\circ$ gives an approximate value for π , which will get closer to π as n gets larger.

If you use your calculator, you will find (for example) that $1000 \sin 0.18^\circ$ is 3.141 587 486 (to nine decimal places). This is quite close to the correct value, which, to 30 decimal places, is:

3.141 592 653 589 793 238 462 643 383 279

There is a pleasant ditty for remembering these figures (should you wish to): count the number of letters in each successive word to obtain the digit in the next decimal place.

Now I, even I, would celebrate
in rhymes inapt, the great
immortal Syracusan, rivaled nevermore,
who in his wondrous lore,
passed on before,
left men his guidance how to circles mensurate.

The 'immortal Syracusan' is Archimedes, as he lived in Syracuse; 'mensurate' is an old word meaning 'measure'.

There is an equivalent definition of π as the *area* of the unit circle (why these two differently specified constants are exactly the same is an interesting question to pursue). A similar method can be used to find an approximation for π based on approximating the *area* of a unit circle by inscribing regular polygons, and noting again that the more sides there are the better the approximation.

► What is the *area* of one of the triangles in Figure 30?

As you saw in Subsection 2.4, the area of *any* triangle is $\frac{1}{2} \times \text{base} \times \text{vertical height}$. So the area of this triangle is $\frac{1}{2} \times 2 \sin(\frac{180}{n})^\circ \times \cos(\frac{180}{n})^\circ = \sin(\frac{180}{n})^\circ \cos(\frac{180}{n})^\circ$.

Therefore the area of the n -sided polygon as a whole is $n \times \text{this area} = n \sin(\frac{180}{n})^\circ \cos(\frac{180}{n})^\circ$. Taking $n = 1000$ gives:

$$\begin{aligned}\pi &\simeq 1000 \sin 0.18^\circ \cos 0.18^\circ \\ &\simeq 3.141 571 983 \quad (\text{to nine decimal places})\end{aligned}$$

This is a slightly worse approximation than using polygonal lengths in the same 1000-gon to approximate the circumference.

From the construction of the regular polygon *inside* the circle, you can see that both approximations (from lengths and from areas) must *underestimate* the actual value of π . Furthermore, it is always the case that $\cos(\frac{180}{n})^\circ < 1$; therefore it is always the case that $n \sin(\frac{180}{n})^\circ \cos(\frac{180}{n})^\circ < n \sin(\frac{180}{n})^\circ$. So the approximation found using the area method is always going to be worse than that found using the length method (though the larger n gets, and hence the closer $\cos(\frac{180}{n})^\circ$ gets to 1, the closer the two approximations will become to each other).

Before leaving Section 3, make sure that you have completed your audit of skills and strategies up to this point in the unit, by filling in the Learning File sheet for Activity 4. In particular, consider how you have been using examples. Have you ever considered what the role of examples is and how you are expected to use them? Mathematics texts frequently include many examples, but it is not always clear what you are expected to do with them. Examples are often used for several purposes, and there is a tendency either to treat them as distractions and pay little attention to them or to try to 'learn' them by rote. Examples can play a key role in demonstrating connections—showing the connection between techniques, results, definitions. But they can also act to introduce an idea or to illustrate the meaning of a result. So whenever you meet an example, it is worth considering what it is exemplifying so that you can make the most use of it.

Outcomes

After studying this section, you should be able to:

- ◇ write down the six trigonometric ratios for an angle in terms of the ratios of pairs of the sides (adjacent, opposite and hypotenuse) of a right-angled triangle, and calculate values for the ratios in specific cases (Activities 15, 16);
- ◇ explain the significance of the concept of similarity for the definitions of the trigonometric ratios, and use these ratios to solve simple practical problems involving right-angled triangles (Activities 17, 18);
- ◇ use the cosine and sine formulas to solve simple practical problems involving triangles (Activities 21, 22, 23);
- ◇ identify trigonometric and other mathematical techniques from written or audiotaped accounts (Activities 19, 24).

4 Spherical trigonometry

Aims This section explores how to calculate distances across the surface of the Earth, and you will see that the results finally quash any hopes of drawing good maps of the Earth's surface. ◇



Trigonometry is all very well for calculating distances and angles over small regions. But what about the distance from (say) Milton Keynes to Singapore? If it were possible to fly from Milton Keynes to Singapore (which of course it isn't, since Milton Keynes does not have an international airport), then by following the shortest route the aeroplane would travel nearly 11 000 km. The question for this section is: how is this distance calculated?

One thing is clear from the outset: it is not the same as the distance from Milton Keynes to Singapore travelling in a straight line. Aeroplanes, after all, do not travel in tunnels—the straight line between these two places goes deeply through the Earth. In fact, the shortest distance is along the so-called 'great circle' route between the two places. But, in order to determine the distance along such a route, it is necessary to define what a 'great circle' is; the discussion will come back to this shortly.

In order to determine intercontinental distances along 'great circle' routes, the fact that the Earth is—approximately—spherical and not a flat plane has to be taken into account. In this section, the Earth will be assumed to be exactly spherical—a modelling assumption that makes the mathematics less complex. Even so, the calculations will be quite demanding. Furthermore, the distance between Milton Keynes and Singapore will be calculated at ground level; an aeroplane flying at a height of 12 000 metres, say, will go a bit further, but the principle is the same—another modelling assumption.

As you may know, aeroplanes go whenever possible by 'great circle' routes. Such routes sometimes take them to somewhat unexpected places: for instance, when going from London to the West Coast of the USA, planes fly over Greenland rather than the Atlantic.

4.1 Calculating lengths on the surface of a sphere

Even if it were just a matter of calculating the length of a straight line, it is not clear how to start until the positions of the two points representing Milton Keynes and Singapore have been specified quantitatively in some way relative to some coordinate scheme. The calculation of the distance along a straight line between two points in the plane is carried out in terms of their (Cartesian) coordinates, by using Pythagoras' theorem. The aim of this subsection is to find a formula for the surface of a sphere analogous to the 'sum of squares' formula given by Pythagoras' theorem. The formula will require some numbers to act on: numbers which specify the positions of the two places in question.

Note that such coordinates in the plane identify single points, whereas cities cover large areas: another simplifying assumption.

- What can play the role for the sphere that Cartesian coordinates play for the plane?

This is a problem that cartographers and navigators solved a long time ago. Latitude and longitude are designed to do just what is required, namely specify the positions of points on the surface of a sphere (thought of as the Earth). Figure 32 reminds you of the definitions of latitude and longitude (it can be easy to forget which is which), which were given in *Unit 6*.

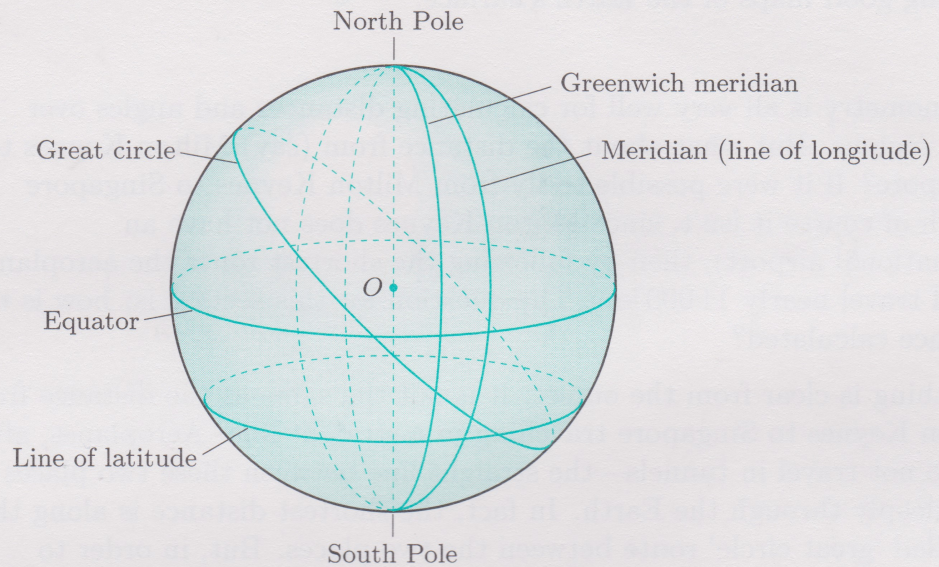


Figure 32 Latitude and longitude

The system is based on drawing imaginary circles on the surface of the sphere. Of the five circles in Figure 32, two are reference circles. These are the Equator and the Greenwich meridian. They play the same role as the x - and y -axes (respectively) for Cartesian coordinates in the plane. Both of these circles are obtained by cutting the sphere by a plane that also passes through the centre O of the sphere. The Greenwich meridian passes through the North and South Poles; it has the (straight) axis through the North and South Poles as a diameter. The Equator is the intersection of the sphere with the plane through its centre that is at right angles to the axis through the North and South Poles.

A *meridian* is a circle on the surface of the sphere obtained by cutting the sphere with a plane containing the Poles (and this plane also necessarily contains the centre O of the sphere). The 'lines' of longitude are all meridians. The Greenwich meridian is, of course, the particular meridian that passes through Greenwich in London—in fact through the observatory there, where you can actually 'see' the meridian in the form of a brass strip inlaid in the courtyard. In some ways, it is a historical accident that Greenwich defines the zero of longitude: in others, it reflects the economic and political power of Britain in the seventeenth century.

A *great circle* is a circle on the surface of the sphere obtained by cutting the sphere with a plane that passes through the centre O of the sphere (but does not necessarily contain the Poles). So all meridians are great circles, but not all great circles are meridians.

Though bear in mind that a meridian is an imagined rather than an actual line.

The 'lines' of latitude are all circles, but apart from the Equator they are not great circles. Their radii decrease as the lines of latitude move nearer the Poles, unlike great circles, whose radii remain fixed—at the radius of the sphere. (Hence, perhaps, the name 'great': they are precisely the circles with the largest radius that can be drawn on the surface of a sphere.)

One thing to bear in mind is that, unlike when using x - and y -coordinates in the plane, this way of specifying coordinates on the sphere is not a uniform system. The two sets of 'lines' making up the grid are not identical to one another from a mathematical point of view.

The coordinates (latitude and longitude) of any point P on the surface of the sphere can be found as illustrated in Figure 33. The latitude of P is the angle that the radius from O , the centre of the sphere, to P makes with the Equatorial plane. The longitude of P is the angle from the Greenwich meridian to the meridian through P measured in the Equatorial plane. An important point here is that, when measuring longitude, only the *half*-meridian from the North Pole to the South Pole through the point being considered is used. So, for example, when measuring longitudes, the Greenwich meridian is always considered to be the *semi*-circle from the North Pole to the South Pole through Greenwich; the other semi-circle, on the other side of the Earth (more or less coinciding with the International Date Line, and shown as a broken line in Figure 33), which completes the (circular) meridian, is not used.

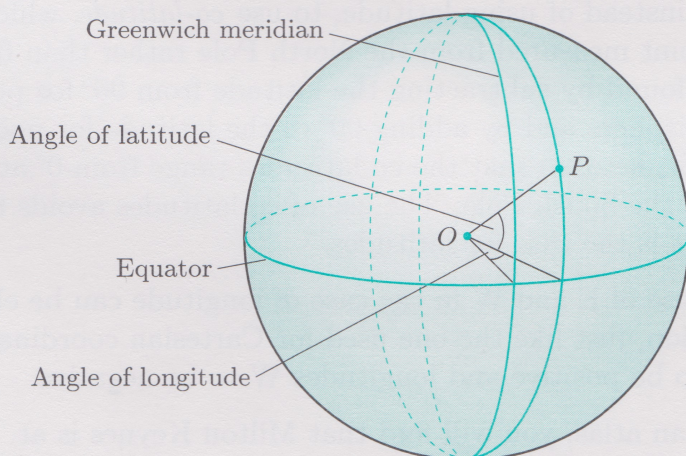


Figure 33

The compass points N, S, E and W are used to specify the directions these angles are measured in: a point in the Northern Hemisphere has latitude so many degrees N, measured *up* from the Equator; one in the Southern Hemisphere has latitude so many degrees S, measured *down* from the Equator. A point to the east of Greenwich has longitude so many degrees E; one to its west, so many degrees W. Longitude ranges from 0° , on the Greenwich meridian, to 180° , more or less on the International Date Line. Points on the Equator have latitude 0° (it is unnecessary to say whether N or S). Latitude ranges from 0° to 90° (which it reaches at the Poles).

Latitude and longitude serve as a system of coordinates for points on the surface of the Earth; and clearly the same principles can be used to develop a coordinate system for any sphere. It is interesting to compare these coordinates with the ones used for the plane. The Equator and the Greenwich meridian serve the same function as the axes do for Cartesian coordinates. From this point of view, the point of intersection of the Equator with the Greenwich meridian functions as the origin.

The net of curves consisting of the meridian 'lines' and the 'lines' of latitude form a grid on the surface of the sphere which in some respects is like the grid of lines parallel to the x -axis and the y -axis for Cartesian coordinates. However, the meridians on the sphere all meet at the Poles, and in this respect at least the properties of the coordinate curves on the sphere are quite different from those of Cartesian coordinate lines in the plane.

There is one further complication, namely that latitudes and longitudes are usually given in degrees and minutes. A *minute* is a subdivision of a degree; there are 60 minutes in a degree. For most computational purposes, it is necessary to convert angles given in degrees and minutes to decimal form. This is done by dividing the number of minutes by 60. From the mathematical point of view, it would be even better to measure the angles in radians; but so long as the problem is seen as one of geography, the angles are likely to be measured in degrees.

It is common, instead of using latitude, to use *co-latitude*, which is the angle of the point measured from the North Pole rather than from the Equator. It is found by subtracting the latitude from 90° for points in the Northern Hemisphere and by adding 90° to the latitude for points in the Southern Hemisphere, so that the co-latitudes range from 0° at the North Pole to 180° at the South Pole. The use of co-latitudes avoids the need to use N and S, as is the case for latitudes.

Similarly, the use of E and W in the case of longitude can be eliminated by a sign convention, just like the one used for Cartesian coordinates: take longitudes E to be positive and longitudes W to be negative.

If you consult an atlas, you will find that Milton Keynes is at ($52^\circ 3' \text{ N}$, $0^\circ 42' \text{ W}$) and Singapore is at ($1^\circ 17' \text{ N}$, $103^\circ 51' \text{ E}$). Since $3/60 = 0.05$, the latitude of Milton Keynes, expressed in degrees as a decimal, is 52.05° N . Its co-latitude is therefore 37.95° , since $90 - 52.05 = 37.95$. The coordinates of Milton Keynes in this mathematically preferred form are (37.95° , -0.70°), while those of Singapore are (88.72° , 103.85°).

The positions of these cities on the spherical model of the Earth are shown in Figure 34. M represents Milton Keynes, S Singapore; O is the centre of the sphere and N the North Pole. The co-latitudes of M and S are marked. Also marked is the angle between the meridians through the two cities, which is 104.55° . It is found by taking the difference between their longitudes, paying due attention to signs. This angle is shown not only on the Equatorial plane but also at the North Pole, where it is shown as the

Latitude is given first. The dash is the symbol for 'minutes'.

angle between the tangents to the meridians through M and S . This is equivalent to the Equatorial representation; but it will turn out to be better when it comes to calculating the distance between M and S . The information in Figure 34 is sufficient for the calculation of the distance between M and S .

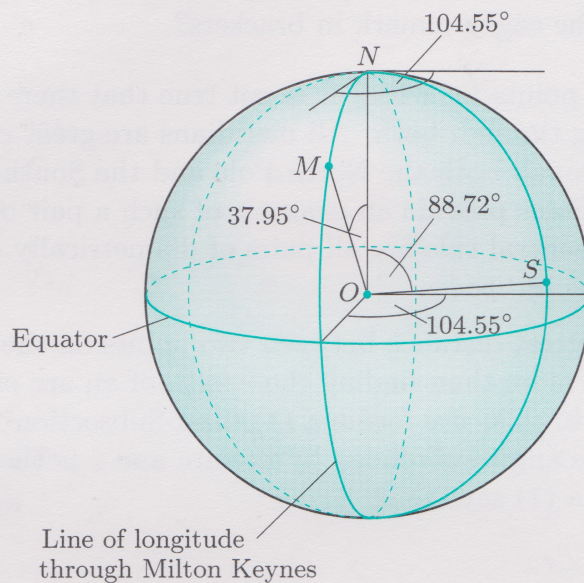


Figure 34

Activity 25

Starting from the latitude and longitude of Milton Keynes and Singapore as it was obtained from the atlas, check the data in Figure 34.

That deals with the problem of finding a way of specifying the positions of the two cities. But the main problem remains, namely how to find the distance between two points on a sphere, where the distance has to be measured over the surface of the sphere. By 'distance' here, think of the *shortest* distance (airlines, remember, choose routes that minimize fuel costs). So the problem becomes one of finding which curve on the surface of a sphere joining two given points has the shortest length, and then determining the length of this curve.

As was indicated earlier, the shortest curve between two points on the surface of the sphere is part of a great circle through the points. Now, given any pair of points on the surface of the sphere (well, almost any pair), there is one and only one great circle passing through them. The shorter of the two circular arcs making up the great circle joining the two points is the curve that has the shortest length of all the curves on the surface of the sphere connecting the two points. For example, the meridian through Milton Keynes is the great circle that joins it to the North Pole. The two points, Milton Keynes (M) and the North Pole (N), divide the great circle into two arcs: one, the longer, passes through the South Pole;

Straight lines play this role in the geometry of the plane: the shortest distance between any two points in the plane is along the unique straight line joining them.

This mention of arcs of circles should remind you of formula (1), developed in Subsection 2.2, which allowed calculation of arc lengths from knowledge of angles.

the other, the shorter, does not. The distance from M to N can therefore be calculated simply by finding the length of the shorter arc along the meridian from M to N .

- ‘Given any pair of points on the surface of the sphere (well, almost any pair), there is one and only one great circle passing through them.’ Why the cagey remark in brackets?

There are pairs of points for which it is not true that there is only one great circle passing through both. All meridians are great circles; all meridians pass through both the North Pole and the South Pole; so the North and South Poles provide an example of such a pair of points. The exceptions to the general rule are all pairs of diametrically opposite points, which are known as *antipodes*.

So, finding the shortest distance between two points on the surface of a sphere involves no more than finding the length of an arc of a circle. And you know how to do this: use formula (1) from Subsection 2.2. If s is the arc length, α is the angle subtended by the arc and r is the radius of the circle, then formula (1) says that:

$$s = \frac{\alpha}{360} \times 2\pi \times r \quad (1)$$

So the problem now becomes one of determining the radius of the great circle and of determining how to measure the angle subtended.

Now, all great circles on the surface of the Earth, modelled as a sphere, have radii equal to the radius of the Earth. So you need to know the radius of the Earth. You have seen one estimate of this: 6112 km (Activity 7). But that is a bit out of date. Here are some more up-to-date figures. The circumference of the Earth measured around the Equator is 40 077 km. The meridional circumference—the circumference measured over the Poles—is 39 945 km. The corresponding radii are therefore 6378.5 km (equatorial) and 6357.4 km (meridional). Using an average (the mean) of these values gives the Earth’s radius as $\frac{1}{2}(6378.5 + 6357.4) \simeq 6368$ km.

Therefore, to find the arc length, you now only need to be able to determine the angle subtended. This, however, is not straightforward—unless both points lie on the same meridian or both points lie on the Equator. For example, consider the distance from Milton Keynes to the North Pole. The angle subtended by the arc at the centre of the circle is known: it is simply the co-latitude of Milton Keynes, namely 37.95° . So, the distance (in kilometres) from Milton Keynes to the North Pole can be obtained, using formula (1), as:

$$\frac{37.95}{360} \times 2\pi \times 6368 \simeq 4218$$

The distance is therefore 4220 km, to three significant figures.

The Earth is not exactly a sphere. As it spins, the Equator is thrown outwards as the centrifugal force effectively reduces the effect of gravity slightly. The Earth is thus flattened slightly at the Poles. Also, the surface of the Earth is not smooth: there are mountains, valleys, and so on. So there can never be an ‘exact’ radius for the Earth.

The answer has been rounded to three significant figures, because both the co-latitude and the radius are accurate to at most four.

Activity 26 *Finding distances on meridians or the Equator*

- (a) London is almost due north of Accra in Ghana (they both have longitude about 0.1°W). The latitude of London is 51.50°N , while that of Accra is 5.58°N . Find the distance between London and Accra.
- (b) Kampala in Uganda and Quito in Ecuador both lie very close to the Equator. (Kampala is actually very slightly north of it and Quito very slightly south, but ignore this for the purposes of the question.) The longitude of Kampala is 32.50°E , while that of Quito is 78.58°W . Find the distance between them.

Ecuador is so named because the Equator runs through it.

It is worth pausing for a moment to compare the method of finding distances along arcs of meridians and of the Equator with the method of finding the distance between any two points in the plane, with Cartesian coordinates (x_1, y_1) and (x_2, y_2) say. Two such points are shown in Figure 35. The distance between the points in the direction of the x -axis is $x_2 - x_1$, and the distance between them in the direction of the y -axis is $y_2 - y_1$. These are the plane analogues of the two distance calculations carried out so far on the sphere: the distance $x_2 - x_1$ is the analogue of the distance along a line of latitude (such as the Equator); the distance $y_2 - y_1$ is the analogue of the distance along a meridian. Now, in order to find the distance between the two points in the plane, the distance in the x -direction and the distance in the y -direction are combined using Pythagoras' theorem, as shown in Figure 35. The rest of this subsection is devoted to finding an equivalent result to Pythagoras' theorem in the case of spheres—that is, to finding a way of combining latitudinal and longitudinal distances to give the distance between two points on a sphere.

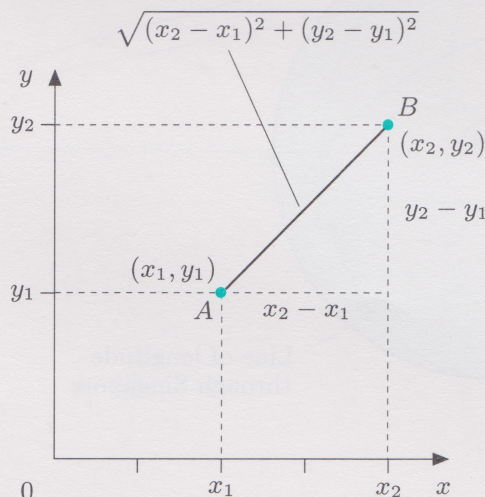


Figure 35

The argument over the next few pages is not easy. Do not worry if you cannot follow it in detail; just concentrate on the general principles. You will not be asked to reproduce it.

To see the way to this result for spheres, consider once more the problem of finding the distance from Milton Keynes to Singapore. The great circle joining these cities is neither a meridian nor the Equator. It is shown in Figure 36.

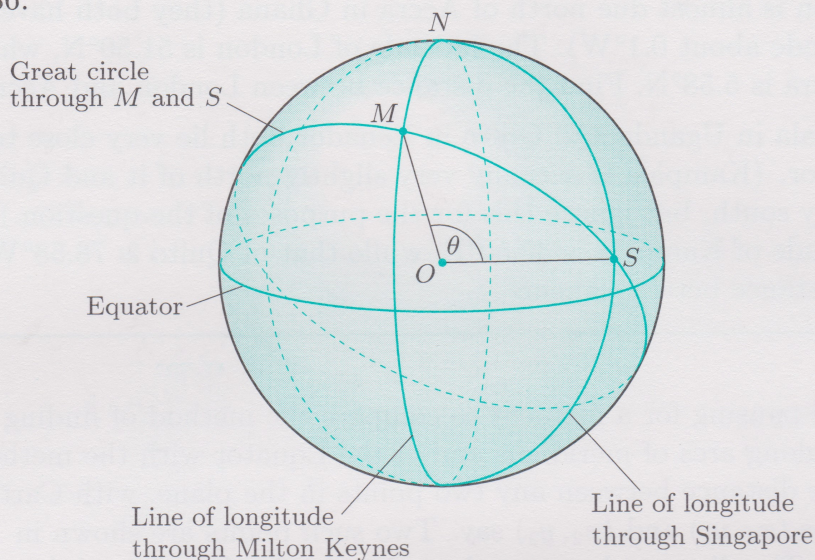


Figure 36 View of the Earth showing the great circle passing through Milton Keynes (M) and Singapore (S)

The calculation of the distance between Milton Keynes and the North Pole drew attention to the fact that all you need to know to find the distance between two points on the Earth's surface is the angle that the arc of the great circle joining them subtends at the centre of the Earth. So finding the distance between M and S involves finding the angle between OM and OS , marked θ on Figure 36.

θ is the Greek letter 'theta', and is commonly used as the algebraic label for an unspecified or general angle.

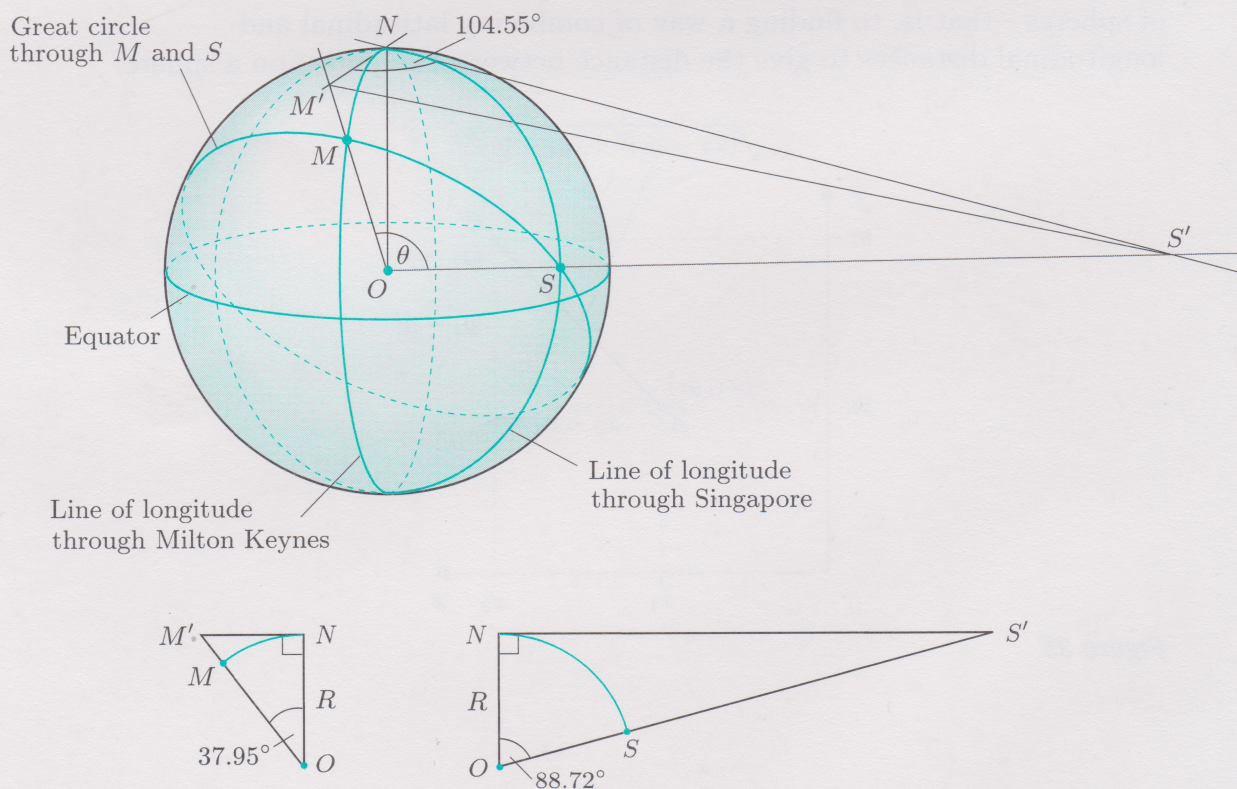


Figure 37 Constructions for calculating θ

Finding θ involves some additional constructions, shown in Figure 37. In Figure 37, the tangent at N to the meridian through M , and the line OM , have both been extended to meet at M' ; likewise, the tangent at N to the meridian through S , and the line OS , have both been extended to meet at S' . In addition, the lines ON and $M'S'$ have been drawn.

The calculation of θ has three main steps. The first involves using the triangles ONM' and ONS' to find the lengths of the lines NM' , NS' , OM' and OS' . The second applies the cosine formula to triangle $NM'S'$ to find the length of $M'S'$. The third applies the cosine formula to triangle $OM'S'$ to find θ .

- 1 The triangles ONM' and ONS' have been drawn separately in Figure 37, so that you can see clearly that they are both right-angled triangles. In each, you know the length of ON (which is the radius of the Earth) and the angle at O (the co-latitudes of Milton Keynes and Singapore respectively). So the lengths of the other sides can be calculated using the tangent and secant of the angles at O .
- 2 In triangle $NM'S'$, you now know—from step 1—the lengths of the sides NM' and NS' ; and you also know the angle at N , which was computed earlier as the angle between the meridians through M and S . So the length of $M'S'$ can be calculated using the cosine formula in the form:

$$c^2 = a^2 + b^2 - 2ab \cos C \quad (8)$$

- 3 In triangle $OM'S'$, you now know the lengths of all three sides—from steps 1 and 2. So the angle θ can be computed using the cosine formula in the form:

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab} \quad (7)$$

Once θ is known, the length of the arc of the great circle between M and S is easily calculated, as before, using formula (1).

The final calculation of θ using formula (1) will necessarily involve the use of a value for the radius of the Earth. It appears at first sight that the radius will also need to be used right from the start of the calculations, in step 1. However, it so happens that all the occurrences of the radius in the three main steps—right up to the calculation of θ —cancel out. Since this makes the calculation easier, no specific value of the radius will be substituted until the last possible moment, but instead it will be represented throughout by the symbol R .

With hindsight, it is possible to see why it is not necessary to know the value of R in order to calculate θ .

► Can you see why?

It arises as another instance of the effects of scaling and similarity. If the whole figure were scaled up or down, the angles would remain unchanged, though the lengths—and in particular the radius—would change by the same scale factor. This means that the value of θ can depend only on the values of the other angles in the figure, and not on the radius of the particular sphere.

Here then is the calculation. (The results of intermediate calculations are given to four significant figures only, to save space, though in fact all calculations have been carried out to the full accuracy of the calculator.)

Calculating the distance between Milton Keynes and Singapore

Step 1

Since triangle ONM' is right-angled, the lengths of NM' and NS' are obtained using the tangent as:

$$\text{length of } NM' = R \tan 37.95^\circ \simeq 0.7799R$$

$$\text{length of } NS' = R \tan 88.72^\circ \simeq 44.75R$$

Similarly, the lengths of OM' and OS' are obtained using the secant as:

$$\text{length of } OM' = R \sec 37.95^\circ = \frac{R}{\cos 37.95^\circ} \simeq 1.268R$$

$$\text{length of } OS' = R \sec 88.72^\circ = \frac{R}{\cos 88.72^\circ} \simeq 44.77R$$

Step 2

The angle at N of the triangle $NM'S'$ is 104.55° , so by the cosine formula in the form of equation (8):

$$\begin{aligned} (M'S')^2 &= (NM')^2 + (NS')^2 - 2(NM')(NS') \cos N \\ &= (0.7799R)^2 + (44.75R)^2 - 2(0.7799R)(44.75R) \cos 104.55^\circ \\ &= [0.7799^2 + 44.75^2 - 2 \times 0.7799 \times 44.75 \times (-0.2512)]R^2 \\ &\simeq 2021R^2 \end{aligned}$$

Therefore:

$$\text{length of } M'S' = \sqrt{2021}R$$

Step 3

Using the cosine formula, in the form of equation (7), on triangle $OM'S'$ gives:

$$\begin{aligned} \cos \theta &= \frac{(OM')^2 + (OS')^2 - (M'S')^2}{2 \times OM' \times OS'} \\ &= \frac{(1.268R)^2 + (44.77R)^2 - 2021R^2}{2(1.268R)(44.77R)} \\ &= \frac{(1.268^2 + 44.77^2 - 2021)R^2}{(2 \times 1.268 \times 44.77)R^2} \\ &= \frac{1.268^2 + 44.77^2 - 2021}{2 \times 1.268 \times 44.77} \simeq -0.1368 \end{aligned}$$

Notice how the R s did cancel out, as predicted.

Using the inverse function $\cos^{-1}$, it follows that:

$$\theta = \cos^{-1}(-0.1368) = 97.87^\circ$$

For the sake of brevity, $M'S'$ is used here instead of 'length of $M'S'$ ', and so on.

Notice in passing that $\cos 104.55^\circ$ is negative: this is because 104.55° is greater than a right angle.

So the distance between Milton Keynes and Singapore (in kilometres) is given, using formula (1), by:

$$\frac{\theta}{360} \times 2\pi \times R \simeq \frac{97.87}{360} \times 2\pi \times 6368 \simeq 10\,878$$

The distance is approximately 10 900 km, to three significant figures.

Activity 27 Air miles (or rather kilometres)

Suppose jumbo jets fly at a steady height of 10 000 metres. How far is it from Milton Keynes to Singapore by air?

You now know how the calculation to find the distance between two points of the surface of sphere can be carried out in any particular case; but it would be better to have a general formula, which could be applied over and over again. Such a formula can be found by carrying out the same sequence of steps in the general case; that is, for a 'triangle' consisting of three points on the surface of a sphere, which are joined in pairs by arcs of great circles, such as the three points M , N and S in Figure 37. Such a 'triangle' is called a *spherical triangle*.

Figure 38 shows a spherical triangle ABC , whose curved sides AB , BC and CA are arcs of great circles of the sphere on which it lies. The point O is the centre of the sphere; OA , OB and OC are the radii from the centre of the sphere O to the three vertices of the spherical triangle. The lengths of the sides of the spherical triangle are determined by the angles they subtend at the centre of the sphere: call these α for the side BC , β for the side CA and γ for the side AB respectively.

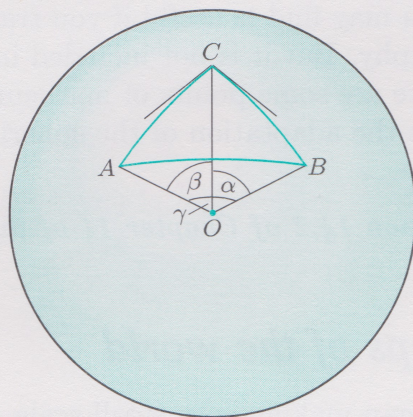


Figure 38 Spherical triangle

Since this is now mathematics rather than geography, think of the angles as being measured in radians. The lengths of the sides are therefore simply $R\alpha$, $R\beta$ and $R\gamma$, where R is the radius of the sphere. The required formula finds γ in terms of α , β and the angle at C between the tangents to the great circles through C that define the sides of the spherical triangle that meet there. From this, the length of the side AB of the spherical triangle is found simply as $R\gamma$. The formula, which is derived in Section 3 of the Appendix, is:

$$\cos \gamma = \cos \alpha \cos \beta + \sin \alpha \sin \beta \cos C \quad (10)$$

Recall from Subsection 2.2 the formula $s = r\alpha$, the equivalent of formula (1) when the angle α is measured in radians, where s is the arc length and r the radius.

This is the *cosine formula* for a spherical triangle. It can be rearranged as:

$$\cos C = \frac{\cos \gamma - \cos \alpha \cos \beta}{\sin \alpha \sin \beta} \quad (11)$$

Note that the derivation of formula (10) does not depend on the angles being measured in radians, and so formulas (10) and (11) are equally valid for angles measured in degrees or in radians.

Activity 28 Distance from Milton Keynes to Rome

The latitude and longitude of Milton Keynes are $52^{\circ}3' \text{ N}$ and $0^{\circ}42' \text{ W}$ respectively; those of Rome are $41^{\circ}54' \text{ N}$ and $12^{\circ}30' \text{ E}$ respectively. Use the cosine formula for a spherical triangle to help you determine the distance from Milton Keynes to Rome.

When the angle at C is a right angle, $\cos C = 0$, and so the cosine formula for a right-angled spherical triangle is simply as follows:

$$\cos \gamma = \cos \alpha \cos \beta \quad (12)$$

This can be thought of as Pythagoras' theorem for the sphere.

Just as the ordinary cosine formula can be programmed on the calculator, the same can be done for the corresponding formula for spherical triangles. With a few adaptations, this becomes a formula for calculating distances along great circle routes between points on the Earth's surface whose latitudes and longitudes are known. The *Calculator Book* gives details of a program to do this: you may find it useful if you travel the world, or if you are interested in geography. But it is not included in the course for its applications alone: there are some points of mathematical interest that arise when carrying out the adaptation of the spherical triangle cosine formula for this purpose.



Now work through Section 14.2 of Chapter 14 of the Calculator Book.

4.2 Making maps of the world

The basic principle of map-making on a small scale—when dealing with a region of the Earth that is small enough to be treated as a plane—is that a figure on the map is similar (in the technical mathematical sense) to the figure on the ground that it represents. That is, angles on the map are the same as the corresponding angles on the ground, and the ratios of lengths on the map are the same as the ratios of the corresponding lengths on the ground. Also, straight lines on the map represent straight lines on the ground, and the ratios of areas on the map are the same as the ratios of the areas of the corresponding regions on the ground. But it is not possible to keep all these desirable properties when trying to represent the whole surface of the (almost) spherical Earth on a plane map.

There is a lot of evidence that this is the case. For example, it was suggested in *Unit 6* that peeling an orange gives an insight into the problem of mapping a sphere. The peel as a whole will not lie flat. By cutting it into smaller pieces you can make it more nearly lie flat; but the pursuit of flatness requires more and more cutting, and the pieces get indefinitely small.

Also, there are some features of the geometry of the sphere that just do not correspond to any features of the geometry of the plane. For example, there are spherical triangles whose angles add up to more than two right angles—indeed, there are even spherical bi-angles, as illustrated in Figure 39.

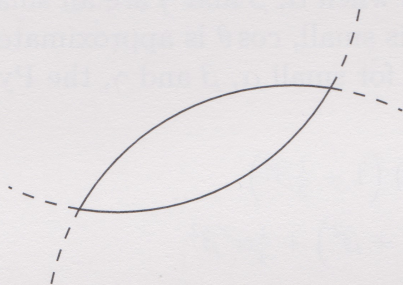


Figure 39 A bi-angle on the surface of a sphere, formed by the intersection of two great circles

Further, there are no similar but not identical triangles on a sphere—if two spherical triangles have the same angles, they are identical to one another. In other words, all similar spherical triangles are congruent.

► What is it precisely that makes it impossible to make an accurate flat map of a sphere? What is it about a sphere that prevents it?

One obvious answer is because a sphere is curved. But this cannot be the whole story, because there are curved surfaces that can be mapped faithfully onto the plane: cylinders and cones are both examples. A demonstration of this property of cylinders and cones is provided by the fact they can be made simply by rolling up pieces of paper, without cutting, stretching or otherwise distorting them.

The investigation of distances over the surface of the Earth in the previous subsection has produced one result that shows quite unequivocally that it is impossible to make a flat map of the surface of a sphere that represents both distances and angles faithfully. If it were possible to make such a map, then the length of the hypotenuse of a right-angled triangle on the map should be proportional to the distance between the two corresponding points on the sphere. This could hold only if the formula for calculating that distance were the same in both cases. But you have seen that the analogue of the Pythagorean formula for the sphere is quite different from the Pythagorean formula for the plane.

Congruence is the mathematical concept of having the same shape and size; if two figures are congruent, then they can be exactly superimposed one on top of the other.

So any method of mapping the whole of the Earth—or any sizable portion of it—must introduce distortions. There are many different ways of mapping the Earth, a number of which were discussed in *Unit 6*. Different ones distort the shapes and sizes of geographical features in different ways. There is no single measure of distortion, and this is one reason why there are so many different ways of mapping the Earth.

However, for small parts of the Earth's surface, the way the Earth's surface behaves geometrically can be closely approximated by a flat map. To see this, consider the Pythagorean formula for the sphere:

$$\cos \gamma = \cos \alpha \cos \beta \quad (12)$$

Consider what happens when α , β and γ are all small relative to the radius of the sphere. When θ is small, $\cos \theta$ is approximately equal to the polynomial $1 - \frac{1}{2}\theta^2$. So for small α , β and γ , the Pythagorean formula for the sphere gives:

$$\begin{aligned} 1 - \frac{1}{2}\gamma^2 &\simeq (1 - \frac{1}{2}\alpha^2) (1 - \frac{1}{2}\beta^2) \\ &= 1 - \frac{1}{2}(\alpha^2 + \beta^2) + \frac{1}{4}\alpha^2\beta^2 \end{aligned}$$

If α and β are small, the term $\frac{1}{4}\alpha^2\beta^2$ is very small, and so can be ignored, giving:

$$1 - \frac{1}{2}\gamma^2 \simeq 1 - \frac{1}{2}(\alpha^2 + \beta^2)$$

Cancelling the 1s and multiplying both sides by -2 gives:

$$\gamma^2 \simeq \alpha^2 + \beta^2 \quad (13)$$

If α , β and γ are measured in radians, the arc lengths a , b and c corresponding to these angles are $R\alpha$, $R\beta$ and $R\gamma$, where R is the radius of the Earth. So:

$$\alpha = \frac{a}{R} \quad \beta = \frac{b}{R} \quad \gamma = \frac{c}{R}$$

Hence formula (13) becomes:

$$\frac{c^2}{R^2} \simeq \frac{a^2}{R^2} + \frac{b^2}{R^2}$$

Multiplying through by R^2 gives:

$$c^2 \simeq a^2 + b^2$$

This is the Pythagorean formula for a plane right-angled triangle. So when α , β and γ are small, the Pythagorean formula for the sphere is a close approximation to the Pythagorean formula for the plane. So, for small distances on the Earth's surface, the way the world's surface behaves geometrically is closely approximated by a flat map.

This was mentioned in Subsection 4.4 of *Unit 13*.



Activity 29 Accuracy of maps

Look back at your Learning File and Handbook notes on maps from *Unit 6*, and add to them notes about the impossibility of drawing accurate flat maps of (large parts of) the Earth's surface and about the possibility of drawing such maps of small parts of the Earth's surface.

This section has made extensive use of diagrams and mathematical examples. You may find it useful to make notes on these as part of your audit of skills and strategies for this section. You have also seen how the conventions of different subject areas—geography and mathematics—differ. Do you feel you are able to switch easily between the two? Using terms/language/symbols appropriately is an important aspect of communicating, and thinking about ideas in different ways can also be helpful as part of learning. You may want to include some notes about this in your Learning File.

Outcomes

After studying this section, you should be able to:

- ◇ use the definitions of latitude, longitude and co-latitude, together with the cosine formula for a spherical triangle, to calculate distances between points whose latitude and longitude are known (Activities 25, 26, 27, 28);
- ◇ explain why it is impossible to draw an accurate flat map of any extensive portion of the Earth's surface without introducing distortion (Activity 29).

5 Geometric form



Aims This section aims to broaden your perspective of the two-dimensional representation of three-dimensional objects via art and architecture. ♦

And geometry. Deep inside us is geometry. My teachers at the university asked us over and over what the reality of geometric concepts was. They asked, Where can you find a perfect circle, true symmetry, an absolute parallel when they can't be constructed in this imperfect, external world?

I never answered them, because they wouldn't have understood how self-evident my reply was, or the enormity of its consequences. Geometry exists as an innate phenomenon in our consciousness. In the external world a perfectly formed snow crystal would never exist. But in our consciousness lies the glittering and flawless knowledge of perfect ice.

(Peter Høeg (1994)

Miss Smilla's Feeling for Snow, Flamingo, London, p. 263)

This section offers another look at representation and geometry, but from a different point of view. Remember the refrain of John and Hope Clearwater from the *Unit 1* reading: 'Cabbages are not spheres ... a fir tree is not a cone ... rivers do not flow in straight lines'. So what can it mean to 'see the world geometrically', whether the natural world or the world of the built environment? Which images do different artists construct to help the viewer see in certain ways, or challenge them to see in a novel or unexpected way? How is someone to draw attention to particular ways of seeing that some or all may have 'lost sight of'?

Figure 40 shows a painting entitled *Self-criticism* by contemporary London-based artist Patrick Hughes, whose work you will see more of later in this section. It draws attention to the fact that, for the door to 'look right' to viewers from *outside* the image, it must fail to function properly *within* the context of the image itself. This provides a gap between what people see it to be and what they know it must be like (for example, rectangular) in order to function as a proper door. The tension between knowledge of three-dimensional objects and conventions surrounding their representation in two dimensions will be explored in this section.

The relationship between what people see and what they know is discussed in *Unit 16*.

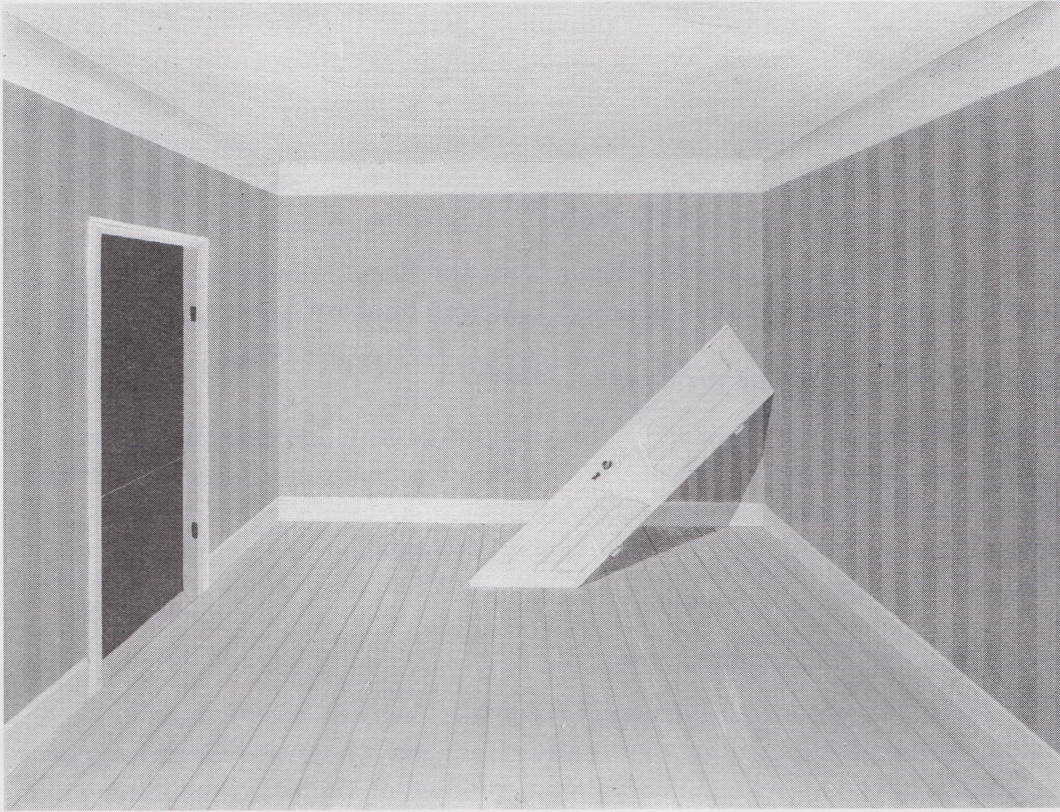


Figure 40 *Self-criticism* by Patrick Hughes

5.1 *Geometry is in the eye of the beholder*

The main activity for this section is to work on a second videotape band, but it is one of quite a different nature from the 'Showing the way they went' band that you viewed at the beginning of this unit. The title of the videotape band is 'Geometric form: in the eye of the beholder'; and, among other things, it offers an experience of seeing the world geometrically. It draws heavily on the 'sight' theme of this final block.

One of the hardest things to try to do is to provide access for someone else to a personal experience, and visual perceptions are extremely individual. Written descriptions are a common means, whether in novels or course units; creating visual sequences is another. But on a printed page, how can authors 'point' to what they are seeing? With a visual sequence, how can a director direct your attention, and say 'see what I see, and in the same way that I see it'?

This videotape band encourages you to think about visual perception. 'Seeing' is a common metaphor in English for 'understanding': 'I see what you mean'. You might care to listen out for further examples of this way of speaking and thinking. One such example is provided by *Unit 1*, where word 'looking' was deliberately chosen: 'looking with numbers', 'looking with relationships', 'looking with symbols', and so on.

The videotape band is designed to be primarily a visual experience: it contains very few words (and all of these occur right at the beginning), so as not to detract from its visual impact. There is a music sound-track, but if you find that this intrudes—as sound too can be made to ‘point’—you could decide to turn the sound down. After you have watched it through once (or twice perhaps, if you find yourself intrigued by some parts of it) and carried out the main activity below, there is a discussion in the next subsection of some of the mathematical themes to which it alludes.

Although there are no tape stops marked, the band falls into three main episodes. The third episode ends with a short final sub-episode.

Episode 1: Looking at pictures

Michelle Selinger (the ‘looker’) is in a studio gallery, looking in turn at three art images from the past. Each in its way has something to offer in terms of seeing geometrically, as does the way Michelle interacts with them, experimenting with ways of seeing. At the end of this episode, she leaves the gallery.

Episode 2: Out and about

Michelle is walking around central Oxford, trying out her geometric ‘vision’, seeing the world geometrically. Her view of central Oxford ignores the details of the architecture and the purposes of the buildings, but instead stresses their shape and form. Stressing and ignoring is the essence of abstraction. What human powers does it take, for instance, in order to look at a building and see a geometric form such as a circle, or a parabola? The camera then takes you further afield to look at other buildings, bridges and constructions from this point of view.

Episode 3: Back in the gallery

This sequence provides three more images: two paintings and a multicoloured poster. It invites questions about the relationship between two-dimensional images and three-dimensional objects, focusing in the middle on different ways of interpreting the ‘same’ drawing of a ‘cube’ and matching it to a ‘real’ cube. It pays attention to different points of view and the possibility of more than one ‘correct’ way of seeing the ‘same’ thing.

Episode 3': Patrick Hughes' work

The videotape band ends with a short sequence filmed in an art gallery, exploring two works of art made by contemporary London artist Patrick Hughes. His work has strong connections with that of artist René Magritte, one of whose pictures appears in the first episode. What are you seeing? What did you think you were seeing?

Activity 30 *Art and architecture on view*

This activity asks you to watch the videotape band. You may care to invite others to watch it with you, so that you can discuss your different visual experiences.

You may care to stop at the end of each episode, as described above, to make notes on what you saw, anything you were puzzled or intrigued by, any questions that came to mind, and any mathematical themes, whether about seeing geometrically or otherwise, that occur to you as worth thinking about. Alternatively, you may decide to watch the whole band through without stopping, and only at the end make detailed notes about what you have experienced and noticed. Your notes could take the form of written words and/or sketches, or you might even like to tape your own commentary.

The images are densely packed, so do not expect to make sense of everything you see at first sitting. You should expect to want to re-view parts or all of the videotape band. You may also like to watch once with the sound on and once with it off.

Now watch band 5 on Videotape 2.



5.2 *What did you see?*

This subsection is structured in terms of the sequence of images on the videotape band. These notes are not intended to be exhaustive: you may well have noticed some of the features commented upon, but also others not mentioned here may have struck you as unusual or noteworthy.

Episode 1: Looking at pictures

The first thing you see is a tower—or perhaps a painting of a tower, as the camera pulls back to reveal a canvas edge—and then another tower; no, a road disappearing into the distance—but it looks identical to the tower. How can a road be a tower, how can the viewer know to see one as a tower and the other as a road? How can the viewer distinguish between the point of the tower—a ‘real’ point in the picture—and the point where the edges of the road apparently meet—a ‘vanishing point on the horizon’? (These ideas are taken up in the next section.)

But then the camera continues to pull back to reveal a picture of a scene that itself has a picture in it. And the inner picture is placed at exactly the right spot and painted at exactly the right size to superimpose precisely over the same image the viewer would see if the picture were not there. Similar figures indeed!

The next picture is a woodcut by Albrecht Dürer: it depicts the very process of making a drawing of a woman, with two grids at work structuring the image, and a sighting device being used to transfer points. What do you notice about the scale of the grids in relation to the distance between them?

The third picture shows three different images of a man (the seventeenth-century French geometer Girard Desargues, who worked extensively on a branch of mathematics which became known as *projective geometry*). The rays of light seem to come from his eye as much as go to his eye; and, where they strike the ground, the lettering is characteristic of a traditional geometric figure.

Projective geometry

Projective geometry takes as its main premise that of a single-eyed observer, and the fact that geometric figures can exactly superimpose on top of one another from this monocular viewpoint. Figure 41 shows two triangles that are 'the same' when considered from the viewpoint (represented by the picture of an eye).

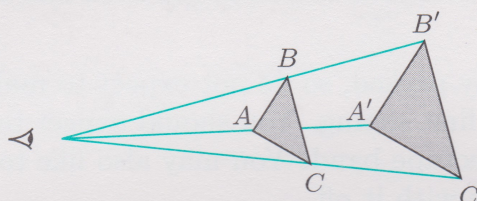


Figure 41 Two triangles in perspective

Such a geometry relies heavily on notions and properties of similar triangles, and reports results that must be true about all figures that are projectively related in this way. Desargues, pictured in Figure 42, was one of the first Western mathematicians to work extensively on such problems.



Figure 42 Girard Desargues

When Michelle looks in turn at these pictures in the gallery, rays emanate from her eye to highlight what she is currently paying attention to. The Ancient Greek geometer Euclid's theory of vision had rays leaving the eye and striking the object, rather than the converse. This seems a very mathematical theory of vision, because there is an important sense in which humans place the forms of geometry onto the world, rather than find them there.

What humans can do is 'see as'. You can see a cabbage *as* a sphere; you can see a bridge *as* an arc of a circle; you can see a church steeple *as* a cone; you can see a reflector dish *as* a parabola. But it is something you are actively doing, superimposing the form (the sphere, the arc, the cone, the parabola) on the exterior world (the cabbage, the bridge, the steeple, the dish), something going *from* your eye *onto* the object. In this way, you can see the world as geometrical. Perceiving a particular real-world object to be a more general, abstract mathematical 'object' is a core ingredient of mathematical modelling.

Episode 2: Out and about

Michelle takes on board this way of 'seeing' and walks around central Oxford construing buildings as geometric forms. The camera then shows different buildings and other human-built structures from around the world, before ending on a blue, glass, squashed cuboid (the technical name for which is *parallelepiped*)—which is revealed after all to be simply a picture in a book.

Activity 31 Seeing geometrically

Sketch some of the shapes you saw in the different buildings. How do they fit together? Walk around your neighbourhood seeing with a geometrical eye. What forms do you see there? Do any questions come to mind? (Recall the mathematical musings from the audiotape band for *Unit 1*—you are being asked for your own geometrical musings here.)

Episode 3: Back in the gallery

Michelle now sees three other images: a multicoloured poster, in which the image portrayed is unclear, and two works of art depicting scenes. She goes from the Van Gogh chair, which looks slightly odd, to a real chair (though made uncharacteristically to have precisely the dimensions of a cube), which is then abstracted into a perfect mathematical cube, of which different views are shown. The animation first shows a sequence of perspectives of a cube depending on the angle and position of the viewer. The 7×7 array only hints at the continuity of change that the viewer is able to handle and say from any view: 'Oh yes, that's a cube'.

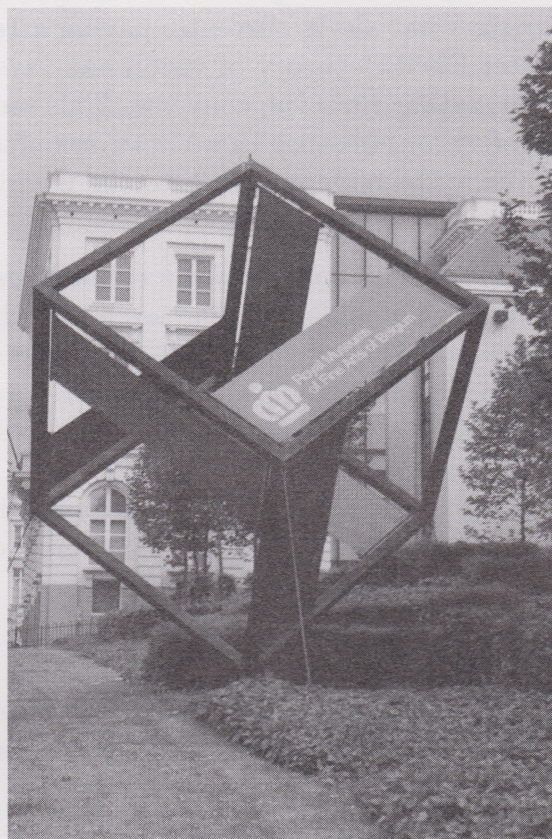


Figure 43 Is this a cube?

The geometric poster ‘becomes’ several different solids before going back into being flat. It is tempting to ask: ‘Which is it *really*?’ Can you see the poster as the primary object and each of the three-dimensional realizations as themselves representations?

Michelle then works from flat representations of the cube to try to locate the correct viewpoint, where what she sees of the ‘real’ cube is identical to the representation she has on her card. The animation suggests something about the way parallel lines in a representation *must* behave in order for it to be a ‘good’ representation. This test criterion is then applied to the Van Gogh chair, and it seems that something is awry.

Activity 32 *When is a chair not a chair?*

- (a) Why does the Van Gogh picture look ‘odd’? How can you tell whether a purported picture of an object can be used to reconstruct the object?
- (b) If you make a perspective drawing of a cube, can you find a point of view (that is, a place to stand) from which the drawing and the image of the cube are indistinguishable? What makes something an accurate drawing of a view of a cube?

Activity 33 Points of view

Consider the geometry of the room you are in. Focus on an object and move around it, drawing different sketches of it from different points of view. Try to focus on what you actually *see* and not what you ‘know’ about the object. Repeat this for other objects. Can you explain why different points of view produce different sketches?

Look at the way walls and ceilings meet at the corners, how doors and windows look. You ‘know’ the top and bottom of a door or window are parallel—but do they *look* parallel from your point of view? (Recall Patrick Hughes’ painting in Figure 40.) Can you explain what you see?

Episode 3': Patrick Hughes' work

One reaction to Patrick Hughes’ work is that it can be hard to tell what is two-dimensional and what is three-dimensional. It can also prove rather unnerving to watch these images.

Standing in front of these two works of art, the viewer is invited to see them as conventional pictures: that is, as flat, two-dimensional, painted images of real scenes. But the moment the viewer’s head moves, something unexpected happens. The image in the picture behaves somewhat like a real-world scene, and not at all like a picture of a real-world scene. If you want to look down an actual hallway, you know that changing your point of view can provide you with a better view. You also know that this is a pointless action when looking at a painting. But with these ‘pictures’, you do seem to see more as you move your head. The image seems to respond to alternative points of view.

The action of the second viewer with the catalogue points up the difference. She was looking at a photograph of the work, which is genuinely two-dimensional. Needless to say, there is no effect of moving the point of view (in her case, by moving the position and angle of the presented image to the eye).

It is only when seen from a very wide angle that the two-dimensional ‘story’ is no longer sustainable and the work can be seen for what it is, namely a deep, three-dimensional, painted sculpture, carefully crafted to look like a two-dimensional painting of a three-dimensional scene.

Hughes made a short television programme about his work for Channel 4. The following excerpt comes from an interview with him in that programme, made as he was constructing a new work using his ‘sculptural’ technique.

One of the important things it is is a perspective drawing—I’ve got to do all the geometry on here [the picture] in order to make the illusion work ... Perspective was meant to be for flat pictures. But I’ve decided to take it and pull it out towards the viewer, take the

The programme was entitled *Artists for Bosnia: a picture for Bosnia* (LWT for Channel 4, 1993) and featured Patrick Hughes and his 1993 picture *The Shadow of War*.

vanishing point by the nose and pull it forward. And once you do that, when people look at it, they thrust it back again, but it starts to move around ...

All these images are framed—they are all framed and constructed in the end, in this modern world, by the cameraman, by the editor, by the technology—and that's very interesting to me because, in my pictures, obviously in anyone's pictures, we're always selecting and cutting and framing and focusing the image ...

Our perspectives on Bosnia are shifting all the time, depending on the news input we get, and we know that the war is changing all the time. The best thing about [my] pictures is they do change—as you move, they change, and as with Bosnia, as our perspectives of the war move, so the war changes, in reality and in our minds.

The Shadow of War now hangs in Glasgow's Gallery of Modern Art. A short note on the painting in a catalogue of some of the gallery's works of art includes the dimensions $85 \times 189 \times 17$ cm (paintings normally have only *two* dimensions) and the following comments from the artist.

This picture was prompted by making a film about the war in Bosnia. My pieces are both made in perspective and made the wrong way around. They are planes which fit together in a rectangle to describe space.

Shadows are helpful allies which differentiate those planes they darken from the ones in the light.

The parts of the buildings—akin to Sarajevo's tower blocks—in the shadow are being destroyed by shelling. Those in the glow of the sun are radiant and have flowering window boxes.

I studied to be a teacher rather than an artist, although I had a show in London the day I ended my studies. Perhaps there is a didactic element in my work. But I hope to let people experience *The Shadow of War*, rather than hector them about it. My work does move and the movement is provided by you.

(Patrick Hughes (1996) in *Gallery of Modern Art, Glasgow: the first years*, Scala Books, London, p. 59)

In another catalogue, covering all of his work using this technique, he adds the following explanation.

I not only make them in perspective, but also I make them in perspective the wrong way round, sticking out [so the points that are physically the closest to the viewer in three dimensions are the points that *within the picture itself* are to be seen as the furthest away from the viewer] ...

It's just a tremendous piece of geometry. It may seem simple in one sense, but it's complex in another. It's a kind of cubist geometry, every plane has to be flat and every plane has to fit and meet with every other plane ...

We live in a rectangular world, and once we see it begin to diminish, we immediately imagine it's disappearing into the distance, we don't imagine it's been constructed eccentrically.

(Patrick Hughes (1994)

Retroperspectives, Flowers East, London, pp. 4, 7, 20)

Geometric language in everyday use

There are a number of examples where, in English, the everyday way of talking about intellectual or cultural or social positions uses geometric ideas and images. Speakers take 'positions', they offer 'perspectives', illustrating 'points of view' and giving 'views' (meaning 'opinions'), following 'lines' of argument and sometimes arguing 'circularly'.

Any object can be seen in an infinite number of ways, and it looks different from each of them. Recall the different ways a simple object like a cube can look, depending on your 'point of view'. (The fact that humans can see 'objects' as single things means in some way they can stress the samenesses and ignore the differences.) This realization can serve as a reminder to beware of those people who claim their viewpoint as privileged, special, or 'the only way to see things': the sequence with the cube can be seen as a metaphor for tolerance.

Finally, there is one more level on which to think about all this. You probably watched the videotape band on a (slightly curved) two-dimensional screen. How aware were you when watching that all the images you were experiencing were flat? How did it affect how you interpreted what you saw as two- or three-dimensional?

5.3 Some concluding discussion

Paintings are not photographs. They are constructed in particular ways and according to quite specific conventions. These conventions involve simultaneous stressing and ignoring what you know to be true about objects in the material world in order to meet (culturally varied) conventions of them 'looking right'.

It is interesting that photographs are now the contemporary standard of 'visual realism'. William Ivins (1969) has written a remarkable book on the history of the making of prints in Europe since 1500. Underlying this theme is an exploration of important differences between words and images for permitting symbolic communication about things. Words written or spoken in different handwriting or different accents are seen as equivalent: 'hand-made pictures, to the contrary, we are aware of as unique things; we see all the differences between them and know the impossibility of repeating any of them exactly by mere muscular action' (p. 159). His recurrent theme is the emergence of the capability for the exact repetition of pictorial statements about things, and the remarkable effect this had on many spheres of knowledge.

Prints and Visual Communication (Da Capo Press, New York).

Ivins points to the influence of the photograph in substantially determining how people come to see.

As the community became engulfed in printed pictures, it looked to them for most of its visual information ... As people became habituated to absorbing their visual information from photographic pictures printed in printers' ink, it was not long before this kind of impersonal visual record had a most marked effect on what the community thought it saw with its own eyes. It began to see photographically, it stopped talking about photographic distortion, and finally adopted the photographic image as the norm of truthfulness in representation. (p. 94)

Thus by conditioning its audience, the photograph became the norm for the appearance of everything. It was not long before men began to think photographically, and thus to see for themselves things that previously it had taken the photograph to reveal to their astonished and protesting eyes. Just as nature had once imitated art, so now it began to imitate the picture made by the camera. (p. 138)

Similar forces are at work with calculator and computer screens. The distortions of the images generated have been discussed in the *Calculator Book*. But what if people's access to mathematical phenomena comes to be primarily through those images: what of screen distortion then?

Recall, from *Unit 6*, Alfred Korzybinski's maxim: 'the map is not the territory'. The picture is not the thing pictured either, but is far more likely to be mistaken for it. The Belgian artist René Magritte painted the picture shown in Figure 44 to comment on this difficulty.

'Ceci n'est pas une pipe' is French for 'This is not a pipe'.



Figure 44 *The Treachery of Images* by René Magritte

This unit started with the trip in the hot-air balloon from which an earlier videotape band was filmed. This allowed the earlier band's viewpoint to be shown and discussed: so often, visual conventions involve masking the viewpoint. One theme that has run throughout this section has been that of multiple perspectives. Mathematically, you have come across this idea before, for example in relation to algebraic expressions. There are many different ways of expressing the same relationships; and by manipulating expressions in certain (usually algebraic) ways, certain novel aspects can be highlighted.

What is seen often depends on finding a particular point of view. Look at the formula for the area of a triangle explored in Subsection 2.4. One-half (base $\times$ height) always gives the same value as (one-half base) $\times$ height, which in turn is always the same as base $\times$ (one-half height). But each different expression arises from a different way of seeing *why* this formula computes the area of a triangle. Mathematical proof is often about showing *why* something has to be the case.

Moving through the world, looking at objects, attempting to produce 'faithful' flat reproductions of objects all raise problems. The problems of artists have mathematical elements, and raise questions about reproduction and representation that lie at the heart of mathematics as well as other disciplines.

Outcomes

After studying this section, you should be:

- ◇ able to comment on what it means to see the world geometrically (Activities 30, 31);
- ◇ alert to some aspects of the two-dimensional representation of three-dimensional objects (Activities 30, 32, 33).

6 Getting things into perspective



Aims This section aims to examine the role of similarity in perspective drawings in art. ♦

This section looks at some geometric aspects of ‘accurate’ perspective as a technique used in much Western representational painting in order that the resulting image ‘looks right’. This process is so common in constructed images in the West that it has become part of the apparatus expected in pictures. The properties of similar triangles can be used to analyse the art of making pictures in perspective: this section is devoted to explaining how.

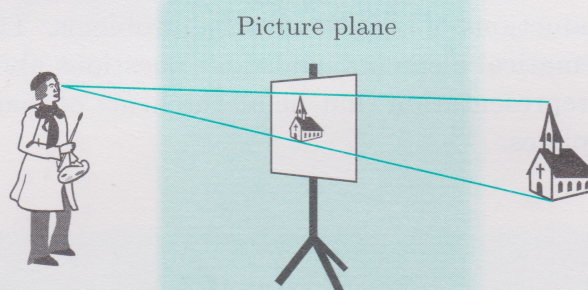


Figure 45 Drawing in perspective

Figure 45 shows an artist drawing a scene in *central* perspective—that is, seen from one fixed point. The position of the artist’s eye—the *viewpoint*—is fixed, as is the picture plane. The *picture plane* is a vertical plane containing the picture. The picture plane is regarded as extending indefinitely in all directions, so that the picture itself occupies only part of it. To draw the scene, the artist has to make a mark on the picture corresponding to each feature of the scene to be portrayed.

- How does the artist determine where in the picture to put the mark corresponding to a given point in the scene?

In principle, what she must do is draw a line from the viewpoint to the point in the scene: where this line intersects the picture plane is where she puts the corresponding mark in the picture. The line represents a ray of light from the point in the scene to the artist’s eye; the points of intersection with the picture plane of the rays of light from any object in the scene to the artist’s eye outline a shape on the picture plane that exactly covers the object. When the finished picture is looked at from the viewpoint, the rays of light entering the viewer’s eye should appear just as if they had come from the object itself.

This point is graphically illustrated by the painting *Euclidean Walks*, by Belgian artist René Magritte, in the opening shot of the videotape band associated with Section 5.

If the same process is carried out for a scene containing a number of objects at different distances from the picture plane, in the resulting picture the representations of those objects should show the variations of size with distance familiar to those with normal vision.

It might be practicable to draw a picture in this way if the artist were drawing on a sheet of glass, but otherwise it is difficult to do because the canvas itself gets in the way. Various ways of carrying out the process in practice were suggested by European artists in the sixteenth and seventeenth centuries, such as the one illustrated in Figure 46, which is a woodcut by the German artist Albrecht Dürer.

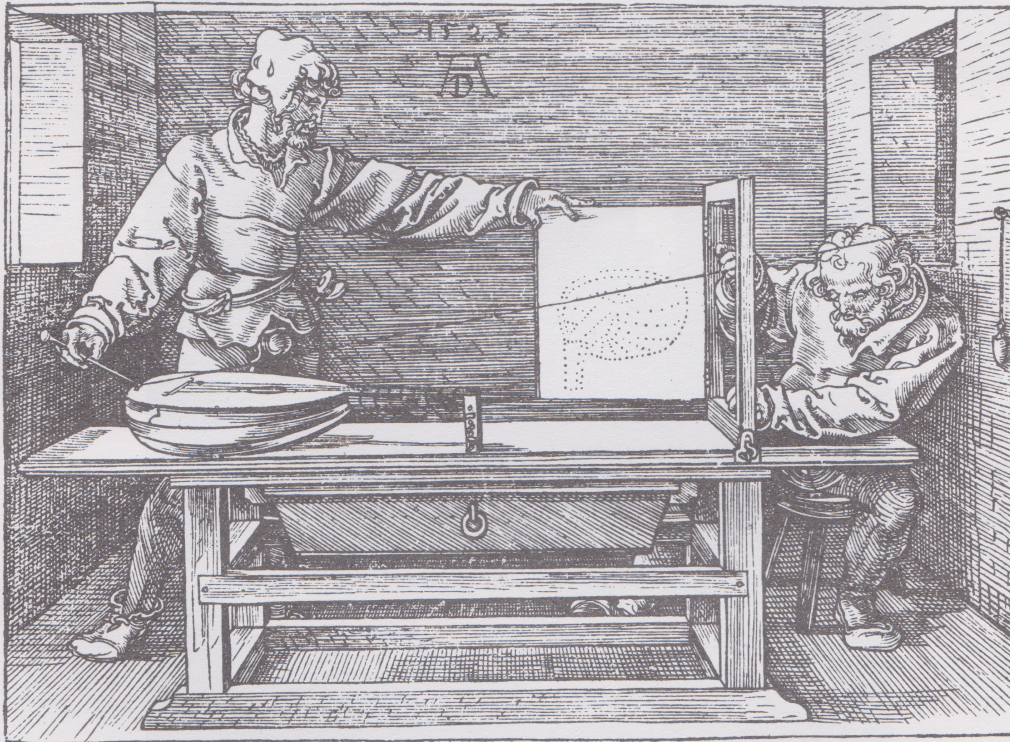


Figure 46 *Man Drawing a Lute* by Albrecht Dürer

A string is stretched from the viewpoint to a point on the object—in this case, a lute (a type of musical instrument); where it meets the picture plane is to be the corresponding point on the picture of the object. The method has been described as follows.

[It] require[s] two operators, one for calculating and one for drawing. A simple pulley is attached to the wall, a piece of string passes through the pulley with a weight at one end and a pointer at the other end. The pulley is substituting for the spectator's eye, the apex of the cone of vision [the viewpoint]. The string from the pulley to the pointer represents a single ray of light and passes through the picture plane. As the man with the pointer fixes different reference points on the lute, his assistant measures off the vertical and horizontal coordinates and plots each new point on the drawing. When there are enough points, he joins the relevant ones, and completes the drawing.

The *apex* is the point at the end of the cone.

(Fred Dubery and John Willats, (1983) *Perspective and Other Drawing Systems*, The Herbert Press, London, pp. 70–1)

Such a picture, painted by Italian artist Crivelli, was also shown on the videotape band.

Many pictures painted in the sixteenth and seventeenth centuries were intended in part to show the painter's mastery of perspective. It is doubtful whether Dürer's laborious method was used in many of these paintings; but they were based on the same principle.

One painter who excelled at the portrayal of interior scenes in perspective was Jan Vermeer. Figure 47 shows one of his paintings.



Figure 47 *The Music Lesson* by Jan Vermeer

The perspective in this painting is so accurate that it has proved possible to reconstruct the actual scene—that is, to work out the dimensions of the room and the positions of the instrument, the player and the music master, as well as the position from which Vermeer painted the picture—from measurements of the painting itself. Just as a map is not the territory, a painting is not the territory either. None the less, on certain pictures painted in perspective, it is possible to take measurements and scale them in order to calculate the corresponding actual distances and angles in the depicted scene. You can read how this is done in one of the reader articles you are asked to study at the end of this section; but first work through the following description of the mathematical theory of central perspective, so that you can make better sense of the article.

- How is the theory of central perspective used to paint a vertical plane parallel to the picture plane, such as the wall opposite the viewer in Vermeer's painting?

In Figure 48, A is the viewpoint, P an actual plane to be painted (called here the *real plane*), and P' the picture plane. Two points B and C , which lie in P , are represented in the picture plane by points B' and C' .

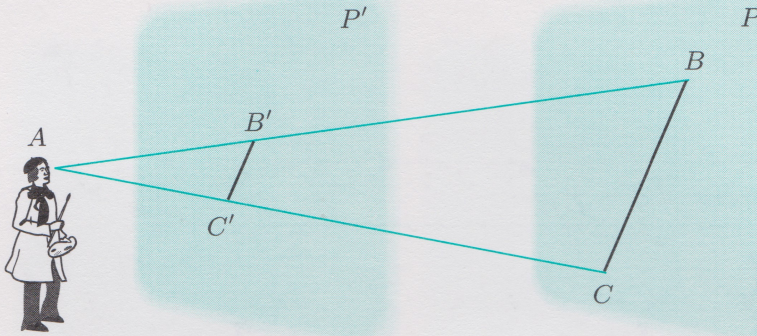


Figure 48

Now the triangles ABC and $AB'C'$ are similar, and so the line BC , whose length is a , is represented in the picture by the line $B'C'$, whose length a' is related to a by:

$$a' = ka$$

Because the triangles are similar, the scale factor k is given by:

$$k = \frac{a'}{a} = \frac{b'}{b} = \frac{c'}{c}$$

So the scale factor k can be found by measuring $c' = \text{length of } AB'$ and $c = \text{length of } AB$, say, and then dividing to get:

$$k = \frac{c'}{c} = \frac{\text{length of } AB'}{\text{length of } AB}$$

Now, if B is a fixed point in the real plane (with corresponding point B' in the picture plane) and if C is *any other* point in the real plane (with corresponding point C' in the picture plane), then the scale factor for $B'C'$ to BC is always given by:

$$k = \frac{\text{length of } AB'}{\text{length of } AB}$$

So the scale factor k is the same for *all* lengths in this real plane.

The scale factor k can be calculated by taking B as the point in the real plane directly opposite the viewpoint, so that B' is the point in the picture plane directly opposite the viewpoint also, as shown in Figure 49 (overleaf). In this case, the length of AB is the perpendicular distance from the viewpoint to the real plane. Call this distance d . And the length of AB' is the perpendicular distance from the viewpoint to the picture plane (in other words, the distance between the painter's eye and the canvas). Call this distance d' . Then

$$k = \frac{d'}{d}$$

Of course, you could equally well have computed k as:

$$k = \frac{b'}{b} = \frac{\text{length of } AC'}{\text{length of } AC}$$

and the relationship between any length a in the real plane and the corresponding length a' in the picture plane is:

$$a' = \frac{d'}{d}a$$

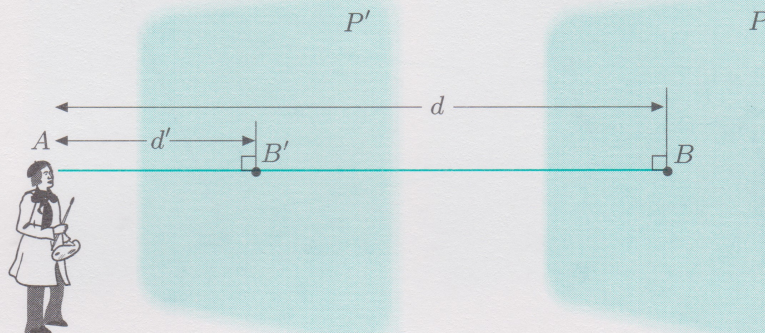


Figure 49

Objects in different real planes parallel to the picture plane will be scaled by different amounts, of course; and objects not lying wholly in such a plane will not be uniformly scaled at all. So more remains to be done to understand perspective.

One profitable way of proceeding is suggested by Dürer's procedure: as you may recall from the description, the role of the assistant is to measure off the vertical and horizontal coordinates of the point in the picture plane and plot each new point on the drawing. This suggests that it would be useful to work out the coordinates of points in the picture plane in terms of the positions of the points they represent.

In order to do this, first choose a pair of axes in the picture plane. One point suggests itself forcibly for the role of origin: the point immediately opposite the viewpoint—that is, the point B' in Figure 49. This point is relabelled with the usual label O for the origin in Figure 50. For the x -axis, take the horizontal line through O in the picture plane; and for the y -axis, choose the vertical line through O . Similar pairs of axes can then be set up in each real plane parallel to the picture plane.

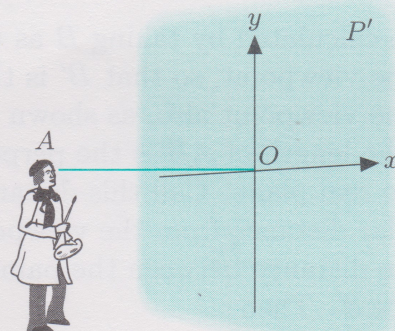


Figure 50 Choosing axes in the picture plane

Consider then a point Q in the scene being painted, which lies in the real plane a distance d from the viewpoint, and whose horizontal and vertical coordinates with respect to the axes in that plane are X and Y , as shown in Figure 51. The coordinates x and y of the corresponding point Q' in the picture plane are easy to calculate: since the picture of the real plane is obtained by scaling all distances with scale factor d'/d , where d' is the distance from the viewpoint to the picture plane, the coordinates of the point Q' are:

$$(x, y) = \left(\frac{d'}{d} X, \frac{d'}{d} Y \right) \quad (14)$$

Note that the point Q is completely general, in that it could be any point with any coordinates X and Y lying in any real plane at any distance d from the viewpoint. So that equation (14) applies for any d , X and Y (and indeed for any distance d' from the viewpoint to the picture plane).

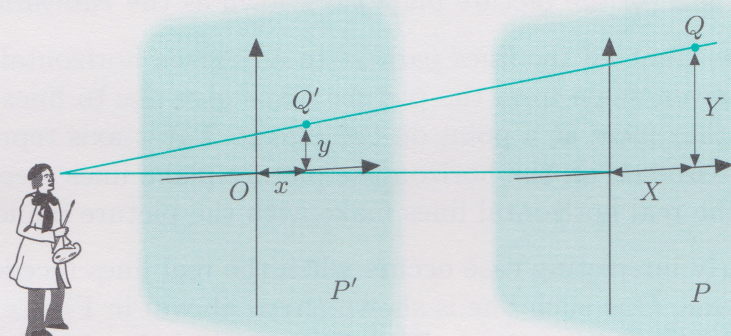


Figure 51 Finding coordinates in the picture plane

With equation (14) to hand, it is possible to explain some of the phenomena of perspective pictures.

One familiar phenomenon is the convergence of receding parallel lines: the way that roads or railway lines seem to meet on the horizon. To see why this is so, first consider lines *perpendicular* to the picture plane. As Figure 52 illustrates, all points on a line that is perpendicular to the picture plane have fixed (that is, constant) values of X and Y ; but, as the points recede from the viewpoint, the value of d increases.

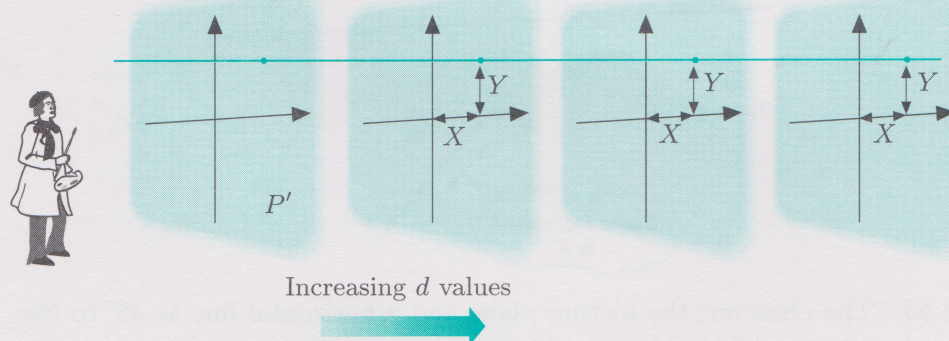


Figure 52

The equation of the corresponding line drawn in the picture plane is obtained by eliminating d'/d between the equations giving the x - and y -coordinates of corresponding points.

$$\begin{aligned} x &= \frac{d'}{d}X, & \text{so that } \frac{d'}{d} &= \frac{x}{X} \text{ (for } X \neq 0\text{).} \\ y &= \frac{d'}{d}Y \end{aligned} \tag{15}$$

Substituting for d'/d in equation (15) gives the corresponding line in the picture plane (for $X \neq 0$) as

$$y = \frac{Y}{X}x = kx, \quad \text{for some constant } k.$$

Remember that X and Y are constants.

Recall the superimposed lines on certain paintings in the videotape band, which all went through one point—or not, in the case of the Van Gogh chair.

This discussion of lines receding at 45° is rather complicated, so do not worry if you cannot follow the mathematical details.

From *Unit 10*, you know this is a straight line that passes through the origin. Thus, the lines in the picture plane that represent lines in space receding perpendicularly from the picture plane should all meet at the origin. The origin (in the picture plane) is known as the *vanishing point*.

It can be shown that all the lines parallel to any given horizontal line receding at a given angle from the picture plane give rise to lines in the picture plane that meet at a point on the x -axis. The x -axis represents the *horizon*. Whereabouts on the horizon the picture plane lines meet depends on the angle the real horizontal lines make with the picture plane.

One particularly interesting case occurs when the real lines recede at 45° to the picture plane. One such line is shown (from above) in Figure 53. For any horizontal line, Y is constant. The slope of a line that recedes at 45° to the picture plane is 1 (since $\tan 45^\circ = 1$), so any increment in d results in an equal increment in X . Therefore the relationship between X and d must take the form $X = d + X_0$ for some constant X_0 (the x -intercept).

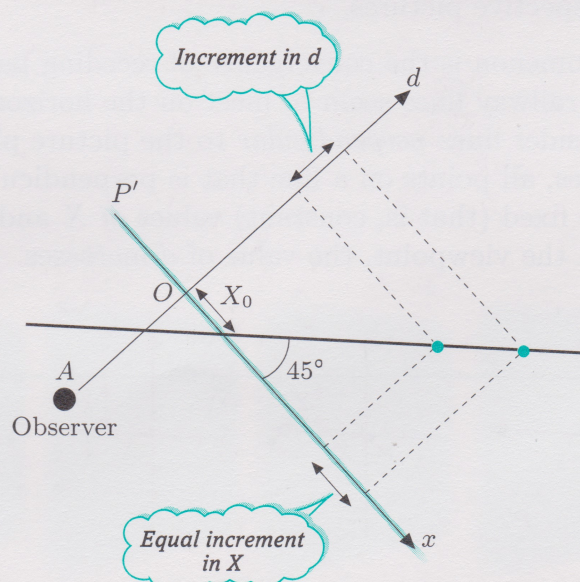


Figure 53 The observer, the picture plane and a horizontal line at 45° to the picture plane, viewed from above

From equation (14), $x = (d'/d)X$, and therefore the coordinates of points on the corresponding line in the picture plane satisfy:

$$x = \frac{d'}{d}(d + X_0) = d' + \frac{d'}{d}X_0 \quad (16)$$

$$y = \frac{d'}{d}Y, \quad \text{so that } \frac{d'}{d} = \frac{y}{Y} \text{ (for } Y \neq 0\text{)}$$

Hence, substituting for d'/d in equation (16) gives

$$x = d' + \frac{X_0}{Y}y = d' + ky, \quad \text{for some constant } k.$$

An important thing to note about this equation is that, when $y = 0$, $x = d'$. So the line in the picture plane meets the horizon (the x -axis) where $x = d'$. But d' is the perpendicular distance from the viewpoint to the picture plane. So the distance from the vanishing point to the point where these picture plane lines (corresponding to horizontals at 45°) meet on the horizon is equal to the distance from the viewpoint to the picture plane. This new point on the horizon is called a *distance point* (there are two distance points, one on either side of the vanishing point).

By using these facts, artists were able to construct perspective pictures without going to the lengths of using a method like Dürer's. The vanishing point and horizon line are drawn first, and the distance points fixed (though these will usually lie outside the picture area itself). Then features like tiled floors can easily be drawn, especially if they are aligned so that the edges of the tiles are parallel and perpendicular to the picture plane. For tiled floors, the vanishing point determines the directions of the edges perpendicular to the picture plane. The distance points can be used to fix the spacing of the tiles, since they will determine the diagonals of the tiles. Tiled floors were popular in pictures of the period when artists were interested in perspective, for just this reason. Vermeer habitually made the construction a little less obvious by setting his tiles at an angle; but, since the edges of his tiles were always at 45° to the picture plane, it made no essential difference.

Remember that X_0 and Y are constants.

You saw a distance point calculated on the Crivelli painting in the videotape band.

Activity 34 Explaining perspective

Write a short explanation of the principles of perspective drawing, using diagrams as well as words if you wish. Include an explanation of why lines of equal length in a drawing do not always represent lines of equal length in reality.

There is a short section in the *Calculator Book* that shows, in a simple case, how changing the relative positions of the viewpoint, the scene being painted, and the picture plane all affect the final picture.



Now work through Section 14.3 of Chapter 14 of the Calculator Book.

You now know enough about perspective to make sense of the two reader articles on Vermeer, his methods and in particular his use of perspective. The second of these shows how the real scene painted by Vermeer in *The Music Lesson* was reconstructed from the painting—something that was possible only because of the accuracy of Vermeer's use of perspective.



Activity 35 *Understanding Vermeer's methods*

Read the reader articles 'Vermeer in perspective' by Jorgen Wadum and 'The photographic accuracy of Vermeer's paintings' by Fred Dubery and John Willats.

Summarize the key points of Vermeer's use of perspective.

You may now like to review parts of the video band for Section 5, looking out for the phenomena and effects discussed in this section.

Outcomes

After studying this section, you should be able to:

- ◇ explain the basic principles of drawing in perspective, with particular reference to the paintings of Vermeer (Activities 34, 35).

Unit summary and outcomes

This unit started in the physical world of maps and navigation, exploring an interest in working out distances and areas, and has ended up with a quite sophisticated mathematical exploration of properties of seeing and representation, both in maps and in paintings. One linking notion throughout has been that of scale and the mathematical property of similarity of plane figures. The named ratios (sine, cosine, tangent, and so on) that comprise the initial subject-matter of trigonometry rely on the similarity properties of right-angled triangles in the plane for their very specification.

The interactive two-way process between thinking about objects and phenomena in the material world and thinking about objects and phenomena of mathematics created in human minds has been exemplified and examined with reference to geometric ideas and imagery. The need to make computations (of distance and area, as well as using numbers to indicate position) has led to a widespread use of arithmetic and algebra. Questions of accuracy, approximation and exactness arise here too, just as they did elsewhere in the course.

Activity 36 A skills audit



Before you finish with this unit, make an assessment of how you feel your study skills have improved since you started the course. In *Unit 5*, you completed a skills audit. Look back to that audit and perform another one now. Were you able to improve and develop those aspects you aimed at? How do you know? Which examples of work could you use as evidence of your improvement? Are there other skills you have developed and improved? In particular, in light of your work on this unit, has your ability to make use of visual images improved?

Being able to monitor and criticize your own learning is an important aspect of being an independent learner. Assessing your own skills, strengths and areas for improvement at regular intervals is one way of monitoring yourself; using feedback from others is another. You have learned that being precise is an important part of communicating mathematically. Are you able to be as precise when you assess yourself? Try to be as clear as you can as you complete your skills audit.

Outcomes

Now you have finished this unit, you should be able to:

- ◇ explain what is meant by the word ‘similar’ when it is applied to pairs of plane figures, and recognize whether or not two figures are similar to each other;
- ◇ explain how to use the properties of similar figures to solve simple geometrical problems;
- ◇ explain the role of similarity in the definition of π , in the definition of the radian measure of angles and in the calculation of lengths of arcs of circles;
- ◇ identify modelling assumptions underlying calculations;
- ◇ follow a proof of Pythagoras’ theorem based on the properties of similar triangles, and explain why the theorem and modifications of it hold;
- ◇ calculate areas of various plane figures;
- ◇ make some observations about the nature and use of geometric diagrams in mathematical argument;
- ◇ write down the six trigonometric ratios for an angle in terms of the ratios of pairs of the sides (adjacent, opposite and hypotenuse) of a right-angled triangle, and calculate values for the ratios in specific cases;
- ◇ explain the significance of the concept of similarity for the definitions of the trigonometric ratios, and use these ratios to solve simple practical problems involving right-angled triangles;
- ◇ use the cosine and sine formulas to solve simple practical problems involving triangles;
- ◇ identify trigonometric and other mathematical techniques from written or audiotaped accounts;
- ◇ use the definitions of latitude, longitude and co-latitude, together with the cosine formula for a spherical triangle, to calculate distances between points whose latitude and longitude are known;
- ◇ explain why it is impossible to draw an accurate flat map of any extensive portion of the Earth’s surface without introducing distortion;
- ◇ have a better idea of what it means to see the world geometrically;
- ◇ be alert to some aspects of the two-dimensional representation of three-dimensional objects;
- ◇ explain the basic principles of drawing in perspective;
- ◇ use visual images appropriately as part of your studying;
- ◇ identify the strengths and weaknesses of your study skills, based on appropriate evidence.

Appendix: Cosine and sine formulas

The material in all three sections of this Appendix is *optional*.

1 The cosine formula for a triangle in the plane

Figure 54 shows a triangle ABC that is of no particular shape (in particular, it is not right-angled). The lengths of the sides a , b and c are assumed to be known. The problem is to find a formula that can be used to determine one of the angles of the triangle, say C , in terms of a , b and c .

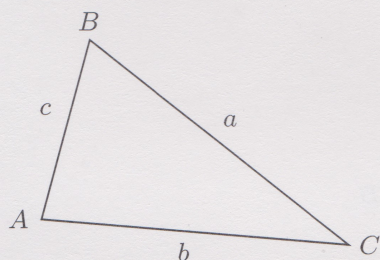


Figure 54

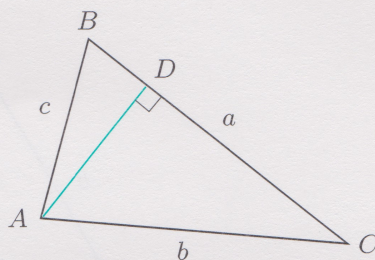


Figure 55

The required formula can be found by drawing the perpendicular line from one vertex (say A) to the opposite side, and call where it meets it D , as shown in Figure 55. Drawing the perpendicular creates two right-angled triangles, ADC and ABD , to which both Pythagoras' theorem and the trigonometric ratios can be applied.

Note first that the length of CD is $b \cos C$. Thus, the length of BD is $a - b \cos C$. Now AD is a side of both of the right-angled triangles ADC and ABD . By using Pythagoras' theorem in each of these triangles in turn, and taking advantage of the occurrence of AD in both to make a bridge between them, an equation involving a , b , c and $\cos C$ can be produced. Call the length of AD x . Then from triangle ADC :

$$x^2 + (b \cos C)^2 = b^2$$

While from triangle ABD :

$$x^2 + (a - b \cos C)^2 = c^2$$

Thus

$$b^2 - (b \cos C)^2 = c^2 - (a - b \cos C)^2$$

since both expressions are equal to x^2 . Expanding the bracket on the right-hand side gives:

$$\begin{aligned} b^2 - (b \cos C)^2 &= c^2 - (a^2 - 2ab \cos C + (b \cos C)^2) \\ &= c^2 - a^2 + 2ab \cos C - (b \cos C)^2 \end{aligned}$$

The terms $(b \cos C)^2$ cancel. The remaining terms can be rearranged to give an equation for $\cos C$:

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

This is the *cosine formula* for the angle at C of a plane triangle. Equivalent formulas exist for the angles at A and B (see Activity 20).

2 The sine formula for a triangle in the plane

If you know the length of one side of a triangle and the two adjacent angles, then evidently you can draw the triangle and measure the other two sides. However, the *sine formula* allows the calculation of the lengths of the other sides without doing any measuring. It can be derived as follows.

Consider the triangle ABC in Figure 56, in which the line CD has been drawn from C perpendicular to the opposite side AB .

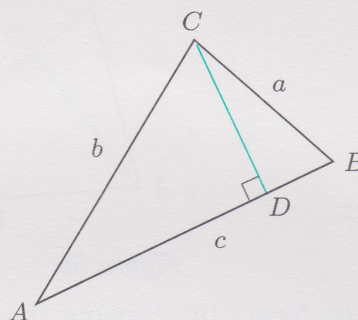


Figure 56

Since the triangle BCD is right-angled, $\sin B = (\text{length of } CD)/a$, so the length of CD is $a \sin B$. The triangle ACD is also right-angled, so by a similar argument the length of CD is $b \sin A$. Since $a \sin B$ and $b \sin A$ are equal to the same thing, they are equal to each other, so $a \sin B = b \sin A$, and so:

$$\frac{a}{\sin A} = \frac{b}{\sin B}$$

A similar argument, involving drawing a perpendicular from B to the opposite side AC , shows that:

$$\frac{a}{\sin A} = \frac{c}{\sin C}$$

Therefore, combining the two equations gives the sine formula:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

So, from the knowledge of two angles and the length of one side of a triangle, the lengths of the other two sides can be obtained from the sine formula. First compute the third angle of the triangle (using the fact that the sum of the angles of any triangle is always 180°). Then, if say the length a of side BC is known, calculate the lengths of AC and AB as:

$$b = \frac{\sin B}{\sin A} a \quad \text{and} \quad c = \frac{\sin C}{\sin A} a$$

3 The cosine formula for a spherical triangle

Figure 57 shows a spherical triangle ABC . AB , BC and CA are arcs of great circles of the sphere, whose centre is at O . OA , OB and OC are the radii to the vertices of the triangle. The sides of the spherical triangle subtend angles α , β and γ at the centre of the sphere; the lengths of the sides are therefore $R\alpha$, $R\beta$ and $R\gamma$, where R is the radius of the sphere. The aim is to find a formula for γ in terms of α , β and the angle at C between the tangents to the great circles that define the sides of the spherical triangle that meet there.

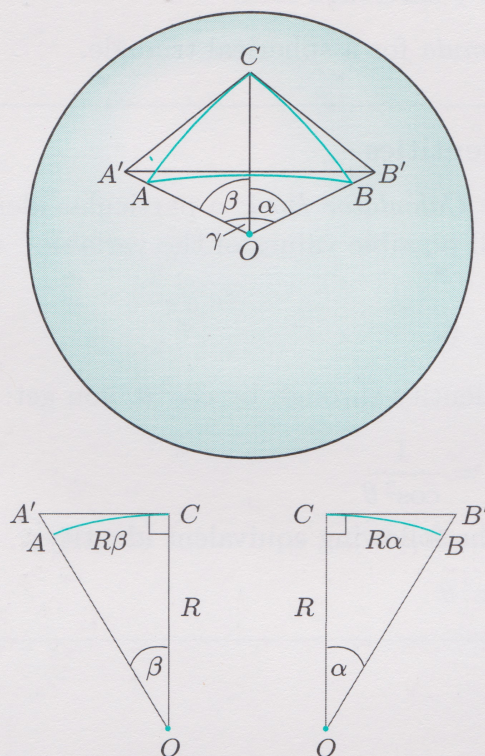


Figure 57

The tangent lines at C to the sides CA and CB have been drawn, projecting out into the space above the surface of the sphere. Call where they meet the radii OA and OB , both extended out into space as well, A' and B' . Then OCA' is a right-angled triangle, with the right angle at C ; and the angle at O is β . Likewise, OCB' is a right-angled triangle, with the right angle at C ; and its angle at O is α . The length of CA' is therefore $R \tan \beta$, and that of CB' is $R \tan \alpha$. Similarly, the length of OA' is $R \sec \beta$, and that of OB' is $R \sec \alpha$.

The cosine formula applied to the plane triangle $B'CA'$ gives the square of the length of $A'B'$ as:

$$R^2 [(\tan \alpha)^2 + (\tan \beta)^2 - 2 \tan \alpha \tan \beta \cos C]$$

Applying the cosine formula again, this time to the plane triangle $A'OB'$, gives:

$$\cos \gamma = \frac{(R \sec \alpha)^2 + (R \sec \beta)^2 - R^2 (\tan^2 \alpha + \tan^2 \beta - 2 \tan \alpha \tan \beta \cos C)}{2R^2 \sec \alpha \sec \beta}$$

$\tan^2 \alpha$ is a shorthand for $(\tan \alpha)^2$.

Notice that the R^2 terms cancel, as they did in the calculation of the distance between Milton Keynes and Singapore.

To simplify this, divide top and bottom by R^2 , then use the identity $\sec^2 \theta = 1 + \tan^2 \theta$ (see the box below), to get:

$$\begin{aligned}\cos \gamma &= \frac{1 + \tan^2 \alpha + 1 + \tan^2 \beta - \tan^2 \alpha - \tan^2 \beta + 2 \tan \alpha \tan \beta \cos C}{2 \sec \alpha \sec \beta} \\ &= \frac{2 + 2 \tan \alpha \tan \beta \cos C}{2 \sec \alpha \sec \beta}\end{aligned}$$

Now $\sec \theta = 1/\cos \theta$, and the 2s cancel, so:

$$\begin{aligned}\cos \gamma &= \cos \alpha \cos \beta (1 + \tan \alpha \tan \beta \cos C) \\ &= \cos \alpha \cos \beta + \sin \alpha \sin \beta \cos C\end{aligned}$$

This is the *cosine formula* for a spherical triangle.

Since $\cos \theta \times \tan \theta = \sin \theta$.

Trigonometric identities

In Chapter 9 of the *Calculator Book*, a particular identity—that is, an equation true for all possible values of the variable—was introduced, namely:

$$\sin^2 \theta + \cos^2 \theta = 1$$

If you divide this identity through by $\cos^2 \theta$, you get:

$$\frac{\sin^2 \theta}{\cos^2 \theta} + \frac{\cos^2 \theta}{\cos^2 \theta} = \frac{1}{\cos^2 \theta}$$

This simplifies to the following equivalent identity:

$$\tan^2 \theta + 1 = \sec^2 \theta$$

Comments on Activities

Activity 1

- (a) There is no comment on this part of the activity.
- (b) The grid references and distances are given in Table 2 on page 54 of *Unit 6*.
- (c) Among the many instances of stressing and ignoring provided by the OS map are that: it stresses boundaries between fields but it ignores their use; it stresses the status of roads (for example, trunk or secondary) but ignores their physical width.

The three-dimensional ridge is indicated by patterns of contours; Mam Tor, Back Tor and Lose Hill are also indicated by patterns of contours; Hollins Cross is indicated by the intersection of several paths.

Activity 2

- (a) It stresses position on the ground and ignores height (and fuel required to gain height). It stresses certain places over which he flew and ignores others. It ignores the magnificent views, the feelings of the people, and so on.
- (b) There is no comment on this part of the activity.

Activity 3

Point 1	Point 2	East	North	Distance as crow flies (km)	Time taken (h)	Speed (km/h) (2 s.f.)
123825	157885	157 – 123 = 034	885 – 825 = 060	$0.1\sqrt{1156 + 3600}$ = 6.896	46/60 = 0.767	9.0
157885	184927	184 – 157 = 027	927 – 885 = 042	$0.1\sqrt{729 + 1764}$ = 4.993	27/60 = 0.45	11
184927	193940	193 – 184 = 009	940 – 927 = 013	$0.1\sqrt{81 + 169}$ = 1.581	9/60 = 0.15	11

Activity 4

You can think of a skill as using a particular technique to solve a mathematical problem, as presenting your work in a particular way, or as a way of reading text for learning. Strategies are more to do with how you learn and so are perhaps more difficult to define clearly. Examples of strategies used in this course include: the modelling cycle; stressing and ignoring; identifying what you know and what you want; looking at generalities and finding particular examples.

Activity 5

- (a) Similar: all circles are the same shape.
- (b) Similar: the major and minor axes determine the shapes of the two ellipses, and they are in the same ratio. (This is tricky if you do not know much about ellipses. Another way of looking at it is that these two ellipses have been obtained by squashing circles by the same factor.)
- (c) Similar: the information given is enough to identify the shapes as squares, and all squares are similar.
- (d) Not similar: the two longer sides of the two rectangles are in the ratio 3 : 2, while the two shorter sides are in the ratio 2 : 1.

- (e) Similar: the angles of the triangles are the same.
- (f) Not necessarily similar: although two pairs of sides are in the same proportion and two corresponding angles are equal, the corresponding angles are not the included angles.
- (g) Not similar: although two pairs of sides are in the ratio 3 : 2, the third pair is not.
- (h) Similar: two pairs of sides are in the same proportion and the included angles are the same.
- (i) Not necessarily similar: except for triangles, polygons can have the same angles but different shapes.

Activity 6

A tree has been modelled by a line and the sun's rays by lines (does light travel in straight lines?). Both tree and stick are (explicitly) assumed to be vertical. It is also assumed the stick can be measured exactly (scaling up even small errors can produce significant errors in the scaled measurement, depending on the size of the scale factor). The tree is assumed to have a single height that does not vary, since the geometric diagram that embodies all of the modelling assumptions is completely static.

Activity 7

The radius of the Earth, according to Eratosthenes, is $38\,400/2\pi \simeq 6112$ km.

There are assumptions about geometrizing the behaviour of the sun's rays as straight lines, about their arrival at the Earth parallel to one another, and about the Earth being spherical.

Activity 8

The original right-angled triangle ABC is completely general; any right-angled triangle could have been drawn and the same constructions and arguments used. Nowhere in the argument has any *particular, numerical, length or relationship* been used—this is often one of the rewards of using algebraic letters for lengths.

Activity 9

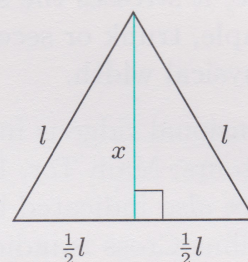


Figure 58

By the symmetry of an equilateral triangle, the perpendicular height cuts the base in half, as Figure 58 illustrates. The equilateral triangle can now be seen as being made up of two right-angled triangles. You know two of the side lengths, and Pythagoras' theorem can be used to find the third—call this length x .

$$l^2 = \left(\frac{1}{2}l\right)^2 + x^2$$

Therefore:

$$l^2 - \frac{1}{4}l^2 = x^2$$

$$x^2 = \frac{3}{4}l^2$$

$$x = \frac{\sqrt{3}}{2}l$$

The area formula for the triangle then gives:

$$\begin{aligned} \text{area} &= \text{base} \times \frac{1}{2} \text{ height} \\ &= l \times \frac{1}{2} \times \frac{\sqrt{3}}{2}l \\ &= \frac{\sqrt{3}}{4}l^2 \end{aligned}$$

Activity 10

Place the parallelogram so that its longest side is horizontal. Cut as indicated in Figure 59, to show that the area of a parallelogram is the same as the area of a rectangle with the same base and same vertical (rather than slant) height.

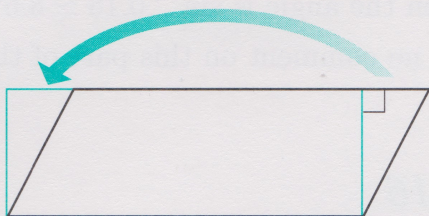


Figure 59

So the area of a parallelogram is given by:

$$\text{area of parallelogram} = \text{base} \times \text{vertical height}$$

Activity 11

- (a) A triangle can be boxed inside a rectangle by drawing in a line through the top vertex parallel to the base and then dropping perpendiculars to the other two vertices from it, as shown in Figure 60. Dropping a perpendicular from the top vertex to the base then produces two smaller triangles, within the original triangle, which together make up the original triangular area. The rectangle is now made up from four triangles. The two on the left are congruent (that is, identical in size and shape). Similarly the two on the right are congruent. So the rectangle formed has twice the area of the original triangle. The area of the rectangle is $\text{base} \times \text{vertical height}$. So the area of the original triangle is one-half ($\text{base} \times \text{vertical height}$).

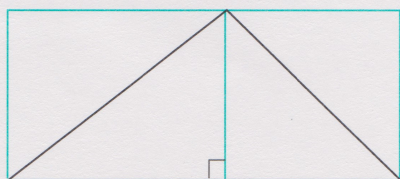


Figure 60

Another way to see this same formula is to rotate the triangle about one of its sides *other than* the base, to produce one of two possible parallelograms, as illustrated in Figure 61. Activity 9 showed that that area of a parallelogram is $\text{base} \times \text{vertical height}$. So the area of a triangle is one-half ($\text{base} \times \text{vertical height}$).

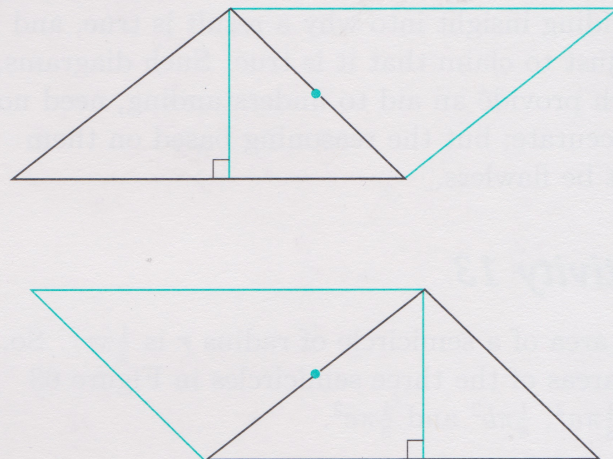


Figure 61

Note that any side of a triangle can be considered as the base, and each has a corresponding vertical height. Can you see why each of these three different sets of base and vertical height must always give the same area for the triangle?

- (c) One way of seeing this is to box the triangle inside a rectangle, as above. The rectangle has the same base and twice the area of the triangle. So the half-rectangle, shown shaded in Figure 62, has the same area as the triangle. And the area of this half-rectangle is $(\text{one-half base}) \times \text{vertical height}$.

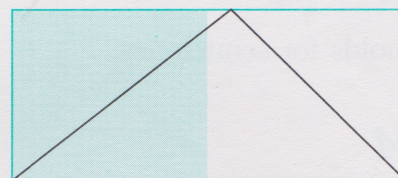


Figure 62

Activity 12

Single geometric diagrams have been used to stand for all triangles or all rectangles or all parallelograms, and then used to support the discussion of a procedure that, it is claimed, can always be carried out, and will always give a particular result.

Geometric diagrams have also been used in providing insight into why a result is true, and not just to claim that it is true. Such diagrams, which provide an aid to understanding, need not be accurate; but the reasoning based on them must be flawless.

Activity 13

The area of a semicircle of radius r is $\frac{1}{2}\pi r^2$. So, the areas of the three semicircles in Figure 63 are $\frac{1}{8}\pi a^2$, $\frac{1}{8}\pi b^2$ and $\frac{1}{8}\pi c^2$.

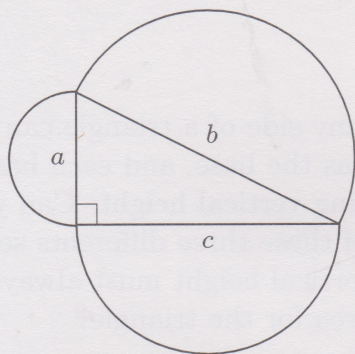


Figure 63

Pythagoras' theorem (for squares) says that:

$$a^2 + b^2 = c^2$$

Multiplying through by $\frac{1}{8}\pi$ gives:

$$\frac{1}{8}\pi a^2 + \frac{1}{8}\pi b^2 = \frac{1}{8}\pi c^2$$

So the result holds for semicircles.

Activity 14

There are no comments on this activity.

Activity 15

- The mathematical gradient is $\tan 15^\circ \simeq 0.27$; that is, about 27%. The road gradient is $\sin 15^\circ \simeq 0.26$; that is, about 26%.
- If the mathematical gradient is 15%, then the angle the road makes with the horizontal is $\tan^{-1} 0.15 \simeq 8.5^\circ$. If the road gradient is 15%, then the angle is $\sin^{-1} 0.15 \simeq 8.6^\circ$.
- There is no comment on this part of the activity.

Activity 16

- $$\begin{aligned}\sin P &= \frac{3}{5} \\ \cos P &= \frac{4}{5} \\ \tan P &= \frac{3}{4} \\ \operatorname{cosec} P &= \frac{5}{3} \\ \sec P &= \frac{5}{4} \\ \cot P &= \frac{4}{3}\end{aligned}$$
- $$\begin{aligned}\cos Q &= \frac{3}{5} \\ \operatorname{cosec} Q &= \frac{5}{4} \\ \cot Q &= \frac{3}{4} \\ \sec Q &= \frac{5}{3} \\ \sin Q &= \frac{4}{5} \\ \tan Q &= \frac{4}{3}\end{aligned}$$
- If you calculate any of $\sin^{-1} \frac{3}{5}$, $\cos^{-1} \frac{4}{5}$ or $\tan^{-1} \frac{3}{4}$, you will get 36.869 897 65. Thus the angle at P is about 37° .
If you calculate any of $\sin^{-1} \frac{4}{5}$, $\cos^{-1} \frac{3}{5}$ or $\tan^{-1} \frac{4}{3}$, you will get 53.130 102 35. So the angle at Q is about 53° . But you could use the fact that the angle at Q is complementary to the angle at P ; and so the angle at Q is $90^\circ - 37^\circ = 53^\circ$.

Activity 17

The points are shown as the vertices of right-angled triangle in Figure 64.

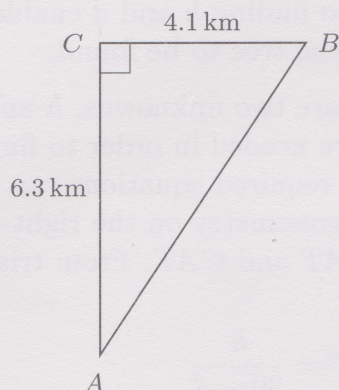


Figure 64

The size of the angle at A in the triangle is:

$$\tan^{-1}\left(\frac{4.1}{6.3}\right) = 33.055\,822\,81^\circ$$

Since C is due north of A , and B is due east of C , this means that the bearing of B from A is roughly 33° .

The distance from A to B can be found by calculating either $4.1 \operatorname{cosec} 33^\circ = 4.1 / \sin 33^\circ$ or $6.3 \sec 33^\circ = 6.3 / \cos 33^\circ$, and comes to 7.5 km to two significant figures. (There are small differences between the values of $4.1 \operatorname{cosec} 33^\circ$ and $6.3 \sec 33^\circ$, but if the angle at A is taken to the full 10-figure accuracy to which it was first given, the answers are identical at 7.516 648 189.) According to Pythagoras' theorem, the distance (in km) is:

$$\sqrt{4.1^2 + 6.3^2} \simeq 7.516\,648\,189$$

Activity 18

- (a) There is clearly a flaw in the question: it fails to say whether or not the tree was vertical. In the absence of any other information, it seems best to assume that it was. This assumption leads to the diagram in Figure 65.

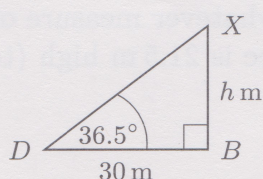


Figure 65

The tree's height, h metres, is given by:

$$h = 30 \tan 36.5^\circ = 22.198\,832\,25$$

The tree was therefore approximately 22.2 m high.

- (b) The information this time leads to the diagram in Figure 66.

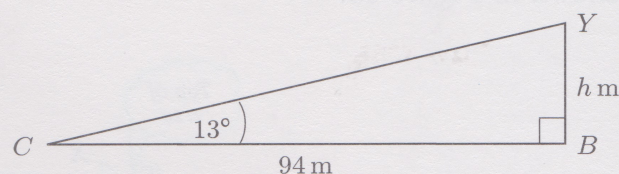


Figure 66

This time:

$$h = 94 \tan 13^\circ = 21.701\,609\,97$$

According to this calculation, the tree was about 21.7 m high.

The tree appears to have grown by about half a metre between December and March! It is not a criticism of Manchester to say that, in its climate, growth of this amount in three comparatively cold months seems to be most improbable. It was stated that the ground was horizontal; but it has been assumed without justification that the tree was vertical. It seems that it cannot have been.

- (c) First, you need to assume that direction of shadow cast at the zenith on both dates was the same. This being the case, the information in parts (a) and (b) can be combined into a single *plane* diagram. But, to draw this diagram, you need to know whether the tree leans away from or towards the direction of the shadows. Suppose it leans away from the shadows, as shown in Figure 67, where T marks the top and B marks the bottom of the tree.

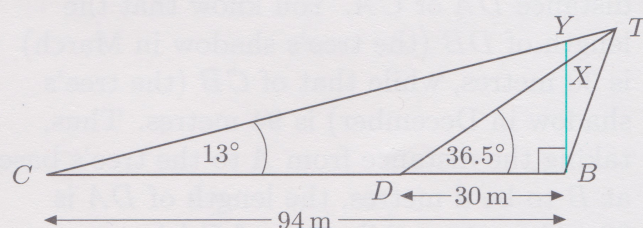


Figure 67

Then the *vertical* heights calculated in parts (a) and (b) correspond to the lengths BX and BY respectively in Figure 67. But in Figure 67, the length of BY is greater than the length of BX ; and this contradicts the values found in parts (a) and (b). Therefore the tree must lean towards the shadow, as shown in Figure 68.

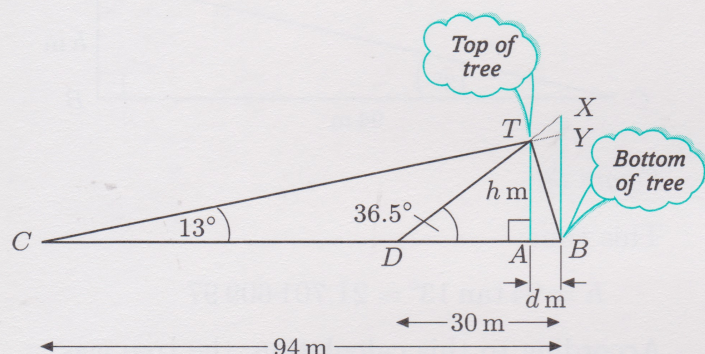


Figure 68

In this figure, the lengths of BX and BY now do correspond to the values found in parts (a) and (b). The point A is vertically below the top of the tree.

The next thing to consider is what exactly is meant by the 'height' of the tree. If the 'height' is the vertical height of the top of the tree above the ground, then all that is required is to compute the length of AT in Figure 68. If, however, the 'height' is the length of the tree's trunk—assuming this to be straight—from the ground to the top of the tree, then the problem is to compute the length of BT in Figure 68.

To find the height, h metres, given by the length of AT , you need to use the formula for the height of a tall object. In order to do this, you need to determine the horizontal distance DA or CA . You know that the length of DB (the tree's shadow in March) is 30 metres, while that of CB (the tree's shadow in December) is 94 metres. Thus, taking the distance from A to the tree's base at B to be d metres, the length of DA is $30 - d$ metres, while that of CA is $94 - d$ metres.

There are now two unknowns, h and d . Notice that once h and d have been found, the trunk height given by the length of BT can be found using Pythagoras' theorem as $\sqrt{h^2 + d^2}$. So finding h and d enables both 'heights' of the tree to be found.

Since there are two unknowns, h and d , two equations are needed in order to find their values. The required equations can be found by using trigonometry on the right-angled triangles DAT and CAT . From triangle DAT :

$$\tan 36.5^\circ = \frac{h}{30 - d}$$

While from triangle CAT :

$$\tan 13^\circ = \frac{h}{94 - d}$$

Rearranging these two equations gives:

$$h = (30 - d) \tan 36.5^\circ$$

$$h = (94 - d) \tan 13^\circ$$

Therefore:

$$(30 - d) \tan 36.5^\circ = (94 - d) \tan 13^\circ$$

Rearranging gives:

$$\begin{aligned} d(\tan 36.5^\circ - \tan 13^\circ) \\ = 30 \tan 36.5^\circ - 94 \tan 13^\circ \end{aligned}$$

And so:

$$\begin{aligned} d &= \frac{30 \tan 36.5^\circ - 94 \tan 13^\circ}{\tan 36.5^\circ - \tan 13^\circ} \\ &= 0.976\,682\,8427 \end{aligned}$$

From the equation $h = (30 - d) \tan 36.5^\circ$, $h = 21.476\,124\,96$, so that the height of the top of the tree above the ground is about 21.5 m.

Using Pythagoras' theorem, the trunk height is:

$$\sqrt{h^2 + d^2} = 21.498\,322\,09$$

So the trunk height is also about 21.5 m.

Therefore, whatever measure of height is used, the tree is 21.5 m high (to one decimal place).

Activity 19

There are no comments on this activity.

Activity 20

$$\cos B = \frac{a^2 + c^2 - b^2}{2ac}$$

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

Activity 21

To find the angle at B in triangle ABD , use the cosine formula:

$$\cos B = \frac{3^2 + 2.72^2 - 2.78^2}{2 \times 2.72 \times 3}$$

$$= \frac{8.67}{16.32} = 0.53125$$

So the angle at B is:

$$\cos^{-1} 0.53125 = 57.91004874 \approx 58^\circ$$

(The same numbers appear in this application of the cosine formula as in the one in the text—as indeed they must, since it is applied to the same triangle. But they appear in different positions in the formula, and sometimes with different signs. This demonstrates how easy it is to make an error when applying the cosine formula, and how important it is to put the right numbers in the right places. The formulation in words—‘the sum of the squares of the lengths of the two adjacent sides minus the square of the length of the opposite side, all divided by twice the product of the lengths of the two adjacent sides’—is very useful in this regard.)

The angle at D is approximately $180 - (56 + 58) = 66^\circ$.

The sizes of the angles of the triangle are all fairly close to each other, and to 60° . But this is to be expected, because the lengths of the sides are all fairly close to each other, which means that the triangle is close to being equilateral. Note that the largest angle is opposite the longest side of the triangle, as you would expect. This provides a useful check of the correctness of the calculations.

Activity 22

A scale plan of the garden, with the angles marked on it, is shown in Figure 69.

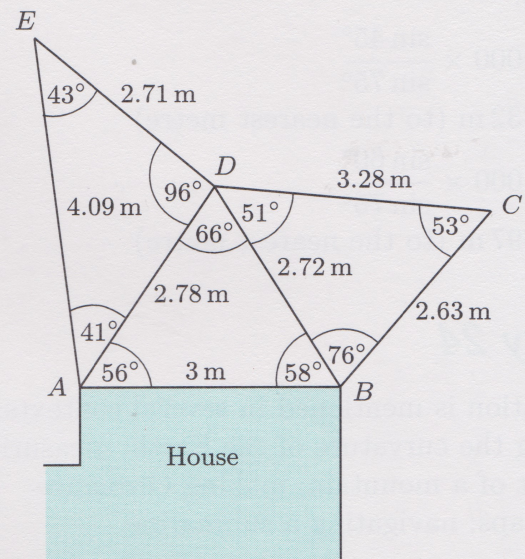


Figure 69

Activity 23

The relative positions of A , B and C and the angles of the triangle ABC are sketched in Figure 70. (The angle at B is $180^\circ - 120^\circ = 60^\circ$; the angle at C is $180^\circ - (45^\circ + 60^\circ) = 75^\circ$.)

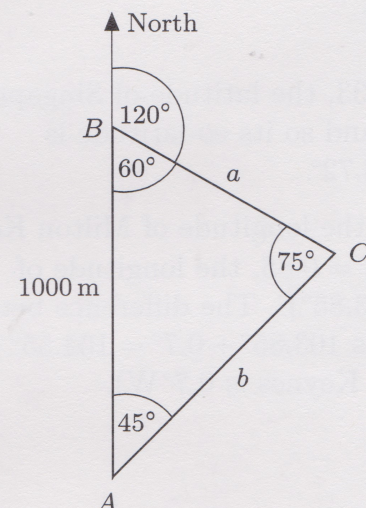


Figure 70

Since the length of AB is $1 \text{ km} = 1000 \text{ m}$, using the sine formula gives:

$$\frac{a}{\sin 45^\circ} = \frac{b}{\sin 60^\circ} = \frac{1000}{\sin 75^\circ}$$

Therefore:

$$\begin{aligned} a &= 1000 \times \frac{\sin 45^\circ}{\sin 75^\circ} \\ &= 732 \text{ m (to the nearest metre)} \\ b &= 1000 \times \frac{\sin 60^\circ}{\sin 75^\circ} \\ &= 897 \text{ m (to the nearest metre)} \end{aligned}$$

Activity 24

Triangulation is mentioned in several contexts: measuring the curvature of the Earth, measuring the height of a mountain; making Ordnance Survey maps; navigating a submarine.

Other concepts mentioned are: data analysis (of response times to fires); digitized maps; acceleration and velocity, and their relationship to position. Bearings and measurements from maps are implicit in all the extracts.

You might like to sketch out the techniques that were explicitly or implicitly used in each extract, and to note down when the cosine or sine formulas might have been used.

Activity 25

Since $\frac{17}{60} \simeq 0.2833$, the latitude of Singapore is about 1.28°N , and so its co-latitude is $90^\circ - 1.28^\circ = 88.72^\circ$.

Since $\frac{42}{60} = 0.7$, the longitude of Milton Keynes is 0.7°W . Since $\frac{51}{60} = 0.85$, the longitude of Singapore is 103.85°E . The difference between the longitudes is $103.85^\circ + 0.7^\circ = 104.55^\circ$ (adding because Milton Keynes is 0.7°W).

Activity 26

- (a) London and Accra have the same longitude, so the shortest distance between them is along the meridian that joins them. The angle subtended by the arc of this great circle is $51.50^\circ - 5.58^\circ = 45.92^\circ$. Therefore the distance between them (in kilometres) is:

$$\frac{45.92}{360} \times 2\pi \times 6368 \simeq 5104$$

So the distance between London and Accra is about 5100 km , to three significant figures.

- (b) Quito and Kampala are both very close to the Equator. So, assuming that they both do lie on the Equator, which is a great circle, the angle subtended by the arc of this great circle is $78.58^\circ + 32.50^\circ = 111.08^\circ$. Therefore the distance between them (in kilometres) is:

$$\frac{111.08}{360} \times 2\pi \times 6368 \simeq 12\,346$$

So the distance between Kampala and Quito is about $12\,300 \text{ km}$, to three significant figures.

Activity 27

It is fine to use the same value for θ , but the value of R must be increased by 10 (since $10\,000 \text{ m}$ is 10 km). So the distance by air is given by:

$$\frac{\theta}{360} \times 2\pi(R + 10) \simeq \frac{97.87}{360} \times 2\pi \times 6378 \simeq 10\,895$$

So the distance is $10\,900 \text{ km}$, to three significant figures, and there is no difference between the surface and the air distances to three significant figures.

Activity 19

There are no comments on this activity.

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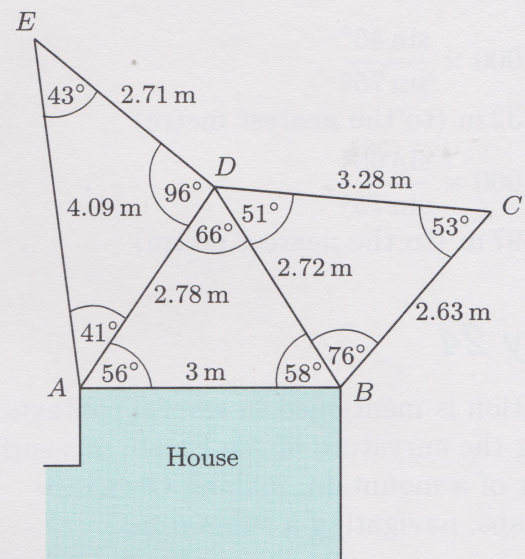


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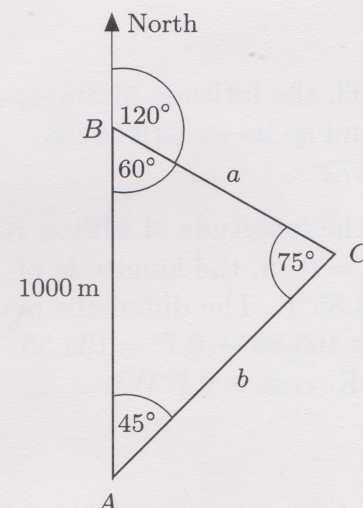


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$$\frac{45.92}{360} \times 2\pi \times 6368 \simeq 5104$$

So the distance between London and Accra is about 5100 km, to three significant figures.

- (b) Quito and Kampala are both very close to the Equator. So, assuming that they both do lie on the Equator, which is a great circle, the angle subtended by the arc of this great circle is $78.58^\circ + 32.50^\circ = 111.08^\circ$. Therefore the distance between them (in kilometres) is:

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So the distance between Kampala and Quito is about 12 300 km, to three significant figures.

Activity 27

It is fine to use the same value for θ , but the value of R must be increased by 10 (since 10 000 m is 10 km). So the distance by air is given by:

$$\begin{aligned} \frac{\theta}{360} \times 2\pi(R + 10) &\simeq \frac{97.87}{360} \times 2\pi \times 6378 \\ &\simeq 10\,895 \end{aligned}$$

So the distance is 10 900 km, to three significant figures, and there is no difference between the surface and the air distances to three significant figures.

Activity 28

You know the decimal forms of the co-latitude and longitude of Milton Keynes already: they are 37.95° and -0.70° . The decimal forms of the co-latitude and longitude of Rome are $90^\circ - 41.9^\circ = 48.1^\circ$ and 12.5° . The angle between their meridians is $12.5^\circ + 0.7^\circ = 13.2^\circ$.

Thus, using formula (10) on a spherical triangle ABC labelled as in Figure 38, with Milton Keynes at A and Rome at B , so that $\alpha = 48.1^\circ$, $\beta = 37.95^\circ$ and the angle at $C = 13.2^\circ$, gives:

$$\begin{aligned}\cos \gamma &= \cos 48.1^\circ \cos 37.95^\circ \\ &\quad + \sin 48.1^\circ \sin 37.95^\circ \cos 13.2^\circ \\ &\simeq 0.972\,255\,989\end{aligned}$$

Therefore the angle subtended by the arc of the great circle through Milton Keynes is:

$$\gamma = 13.527\,934\,45^\circ$$

Hence, since γ is given in degrees, the distance in kilometres between Milton Keynes and Rome is $\frac{\gamma}{360} \times 2\pi \times R$, where $R = 6368$ km is the radius of the Earth. Hence the distance is given by:

$$\begin{aligned}\frac{\gamma}{360} \times 2\pi \times R &= \frac{13.527\,934\,45}{360} \times 2\pi \times 6368 \\ &= 1503.529\,358\end{aligned}$$

So the distance from Milton Keynes to Rome is just over 1500 km.

Activity 29

Your additions to your notes should include the following: that the angles of a spherical triangle generally do not add up to 180° ; that the cosine formulas for planar and spherical triangles are different; that the Pythagorean formulas are different for the plane and the sphere, but that for small regions the spherical version reduces approximately to the planar version.

Activities 30–33

There are no comments on these activities, though some of the issues they raise are discussed in the remainder of this and in the next section.

Activity 34

Your explanation should include some mention of similar triangles and the concepts of picture plane, viewpoint, vanishing point, horizon and distance points. The reason why drawn lines of equal length do not always represent real lines of equal length is that lines different distances from the viewpoint are in different real planes and so the scale factor $k = d'/d$ will vary.

Activity 35

There are no comments on this activity.

Activity 36

There are no comments on this activity.

Acknowledgements

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